Nonlinear drag forces and the thermostatistics of overdamped motion

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Diverse processes in statistical physics are usually analyzed on the assumption that the drag force acting on a test particle moving in a resisting medium is linear on the velocity of the particle. However, nonlinear drag forces do appear in relevant situations that are currently the focus of experimental and theoretical work. Motivated by these developments, we explore the consequences of nonlinear drag forces for the thermostatistics of systems of interacting particles performing overdamped motion. We derive a family of nonlinear Fokker-Planck equations for these systems, taking into account the effects of nonlinear drag forces. We investigate the main properties of these evolution equations, including an H-theorem, and obtain exact solutions of the stretched q-exponential form.

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I. INTRODUCTION

Nonlinear Fokker-Planck equations (NLFPEs) [1] have been recently applied to the study of diverse instances of complex systems [2–9]. Particular applications include type-II superconductors [8,10], granular media [11], and self-gravitating systems [12,13]. An NLFPE governs the behavior of a time-dependent density $\rho(\boldsymbol{r},t)$, where $\boldsymbol{r} \in \mathbb{R}^N$ represents a point in an appropriate N-dimensional configuration space. One of the most intensively studied NLFPE comprises two components: a power-law diffusion term and a linear drift one [14]. In several of the above mentioned applications, the density $\rho(\boldsymbol{r},t)$ has to be interpreted as a physical density corresponding to the distribution of particles in configuration space, and not as a statistical ensemble probability density [8,10,15,16]. In these scenarios, the nonlinear diffusion term constitutes an effective description of the interaction between the particles of the system, while the drift term takes into account the effects of other external forces that act on them. Nonlinear diffusion also occurs in connection with other phenomena [17,18] constituting, for instance, the basis of some relevant phenomenological models for the spread of biological populations [19–22] and for the spread of energy in nonlinear disordered lattices [23].

The nonlinear power-law Fokker-Planck equations exhibit remarkable features that are both physically relevant and mathematically interesting. They obey an H-theorem that can be formulated in terms of a free-energy-like quantity [24], associated with the S_q nonadditive entropic measures [25,26]. These entropic functionals have been recently applied to the study of diverse phenomena in complex systems [27–33]. In some important cases, the nonlinear power-law Fokker-Planck equations have exact analytical solutions of the q-Gaussian shape, that can be interpreted as maximum entropy densities optimizing the S_q measures under appropriate, simple constraints [25,27]. This is, indeed, one of the most important manifestations of the deep connection that exists between the nonlinear Fokker-Planck dynamics and the thermostatistical formalism based on the S_q entropies. Although this connection was discovered more than two decades ago [14], research on its diverse physical implications has only flourished in recent years (see, for instance, Refs. [2,15,34–37] and references therein).

In fact, the S_q-NLFPE connection [14] is central to one of the best understood mechanisms accounting for the successful phenomenological description of various complex systems by the q-thermostatistical theory [27]. The experimental results on granular media recently obtained by Combe *et al.* [11] constitute a notable achievement along these lines. These authors verified, within a 2% error and for a wide range of the relevant experimental settings, a scale relation derived theoretically in 1996 from the maximum S_q entropy solutions of the power-law NLFPE [38]. This is arguably one of the most remarkable quantitative predictions that has been made using the thermostatistical theory based on the S_q entropies.

In most applications of the power-law NLFPEs to study the thermostatistics of systems of short-range interacting particles, performing overdamped motion under an external confining potential, it is assumed that the drag force acting on the particles depends linearly on velocity. This force is due to the interaction between the particles constituting the system under consideration and the medium in which these particles are moving. Motivated by recent experimental and theoretical developments, concerning the drag forces acting on test particles moving in different types of media [39–41], we will explore in the present work the effects that nonlinearities in the drag force have on the thermostatistics of the aforementioned types of many-body systems. It has long been known that other types of velocity dependence for drag forces, beyond the linear one, do occur in nature. The theoretical study of the dynamics of particles moving under different types of drag forces goes back, at least, to Newton's Principia (for a nice discussion, see Chapters 19 and 20 of Ref. [42]). However, in recent years there have been remarkable advances, both at the theoretical and at the experimental levels, in the detailed microscopic understanding of the drag forces on test particles moving in different environments [39–41]. These investigations allowed the characterization of the types of interactions between test particles and environment particles, that lead to specific departures from linear drag. In this work we consider the thermostatistics of systems of confined interacting particles performing overdamped motion under nonlinear drag forces. We derive an associated NLFPE and investigate its physically relevant properties. In particular, we establish its stationary-state solution and prove an H-theorem obeyed by a free-energy functional that involves the entropy S_q. Moreover, we obtain particular, semianalytic time-dependent solutions and explore their main features.

II. THE NONLINEAR POWER-LAW FOKKER-PLANCK EQUATION

The NLFPE with a power-law diffusion term reads

$$\frac{\partial \rho}{\partial t} = \mathcal{D}\nabla^2\left[\rho\left(\frac{\rho}{\rho_0}\right)^{1-q}\right] - \nabla\cdot(\rho\boldsymbol{K}), \tag{1}$$

where $\mathcal{D}$ stands for the diffusion constant with $[\mathcal{D}(2-q) > 0]$, $\boldsymbol{K}(\boldsymbol{r})$ denotes a drift force, and q is a dimensionless parameter associated with the power-law nonlinear diffusion term. The time-dependent density $\rho(\boldsymbol{r},t)$ and the constant ρ_0 have dimensions of inverse (N-dimensional) volume. Equation (1) is sometimes written in the guise $\frac{\partial P}{\partial t} = \mathcal{D}\nabla^2(P^{2-q}) - \nabla\cdot(P\boldsymbol{K})$, where $P(\boldsymbol{r},t) = \rho/\rho_0$ is a dimensionless quantity.

In the present work, we shall consider only drift fields that are determined by the gradient of a potential function $U(\boldsymbol{r})$, $\boldsymbol{K}(\boldsymbol{r}) = -\nabla U$ (that is, we are not going to consider curl forces). We follow here the usual convention in the literature on the Fokker-Planck equation, and call $\boldsymbol{K}$ a "force" and U a "potential," even though $\boldsymbol{K}$ does not have dimensions of force, nor does U have dimensions of energy. As we shall see in Sec. III, these two quantities are, however, respectively proportional to a force and a potential energy with the correct dimensions. The power-law NLFPE admits the q-exponential stationary solution [27],

$$\begin{aligned}\rho_q(\boldsymbol{r}) &= \rho_0 A \exp_q[-\beta U(\boldsymbol{r})]\\ &= \rho_0 A[1-(1-q)\beta U(\boldsymbol{r})]_+^{\frac{1}{1-q}},\end{aligned} \tag{2}$$

where A and β are positive constants satisfying $(2-q)\beta\mathcal{D} = A^{q-1}$, and the q-exponential function, $\exp_q(x) = [1+(1-q)x]_+^{\frac{1}{1-q}}$, vanishes for $1+(1-q)x \leqslant 0$. We assume that the potential function $U(\boldsymbol{r})$ is bounded from below so that, choosing appropriately the zero of energy, the minimum value adopted by $U(\boldsymbol{r})$ is $U_{\min} = 0$, and $U(\boldsymbol{r}) \geqslant 0$. We also assume that the shape of $U(\boldsymbol{r})$ leads to a stationary distribution $\rho_q(\boldsymbol{r})$ of finite norm, $\int \rho_q(\boldsymbol{r})\, d^N\boldsymbol{r} = I < \infty$. We do not require that $I = 1$, because in many applications ρ represents a physical density (as opposed to a probability one). The allowed range of q values yielding stationary densities ρ_q, with finite norm, depends on the form of the potential function $U(\boldsymbol{r})$. The stationary density $\rho_q(\boldsymbol{r})$ coincides with the distribution that optimizes the q entropy

$$S_q[\rho] = \frac{k}{q-1}\int \rho\left[1-\left(\frac{\rho}{\rho_0}\right)^{q-1}\right] d^N\boldsymbol{r}, \tag{3}$$

under the constraints associated with the norm and the mean value $\langle U\rangle = \int \rho\, U d^N\boldsymbol{r}$ of the potential [14,27]. The positive constant k determines the dimensions of S_q, as well as the units in which this quantity is measured. When $q \to 1$ the NLFPE reduces to the standard linear Fokker-Planck equation, $\frac{\partial\rho}{\partial t} = \mathcal{D}\nabla^2\rho - \nabla\cdot(\rho\boldsymbol{K})$. In this limit, the stationary solution (2) reduces to the well-known exponential one, $\rho_{\mathrm{BG}}(\boldsymbol{r}) = A\exp[-\beta U(\boldsymbol{r})]$, with $\beta\mathcal{D} = 1$.

As already mentioned, the power-law NLFPE satisfies an H-theorem formulated in terms of a free-energy-like quantity, associated with the S_q entropic measures [24]

$$\frac{d}{dt}[\langle U\rangle - (\mathcal{D}/k)S_{q^*}[\rho]] \leqslant 0. \tag{4}$$

The functional S_{q^*} corresponds to the index $q^* = 2-q$. For integer values of the entropic parameter q, the power-law NLFPE can be written in a simpler form by absorbing the factor ρ_0^{q-1} into the diffusion constant. That is, redefining the diffusion constant as $D = \mathcal{D}\rho_0^{q-1}$, the NLFPE can be recast under the guise

$$\frac{\partial\rho}{\partial t} = D\nabla^2(\rho^{2-q}) - \nabla\cdot(\rho\boldsymbol{K}). \tag{5}$$

It should be emphasized that, even though (5) has the form of the NLFPE for a dimensionless density, the quantity ρ appearing in this equation still has the dimensions of inverse volume.

III. POWER-LAW DRAG FORCES AND NONLINEAR FOKKER-PLANCK EQUATIONS

It was shown in Ref. [8] that the power-law NLFPE can be regarded as governing the evolution of the spatial density of a system of particles with short-range interactions that perform overdamped motion under the effect of an external confining potential $W(\boldsymbol{r})$ (as we shall briefly explain below, this potential W is proportional, but not identical, to the potential U appearing in the NLFPE). This interpretation of the power-law NLFPE turned out to be very fruitful and led to a considerable amount of further investigations on the thermostatistics of the above-mentioned many-body systems. It is crucial to the theoretical developments reported in Ref. [8], and to all the subsequent works along these lines, to assume a linear dependence of the drag forces on velocity. This state of affairs raises a natural question: Is it possible to extend the approach advanced in Ref. [8] to scenarios involving nonlinear drag forces? To address this question is the the principal aim of the present contribution.

We shall consider a system constituted by particles of mass m that move in an N-dimensional space and interact via short-range, repulsive forces. These particles are also under the effects of an external confining potential W and of a drag force due to a resisting medium. Therefore, there are three contributions to the total force acting on one of these particles. These contributions are given by the force arising from the interaction with the other particles of the system, the force derived from the potential W, and the nonlinear drag force

$$\boldsymbol{F}_{\text{drag}} = -\alpha\, \dot{\boldsymbol{r}} \left| \frac{\dot{\boldsymbol{r}}}{v_0} \right|^{\lambda}, \tag{6}$$

characterized by the constants $\alpha > 0$, $v_0 > 0$, and the exponent $\lambda > -1$. The condition $\lambda > -1$ is necessary for having a drag force whose modulus is an increasing function of the modulus of the particle's velocity. The fact that the interaction between the particles of the system is repulsive and has a short-range implies that, at a given time, each particle only interacts with those particles located within its immediate neighborhood. This intuitive idea can be put in a precise quantitative form, as we now briefly explain (see Ref. [8] for more details). Let $\mathcal{F}(|\boldsymbol{r}' - \boldsymbol{r}|) \geqslant 0$ be the strength of the force felt by a particle located at $\boldsymbol{r}$ due to another particle at $\boldsymbol{r}'$. Since this force is repulsive, its vector representation is $\mathcal{F}(|\boldsymbol{r}' - \boldsymbol{r}|)(\boldsymbol{r} - \boldsymbol{r}')/|\boldsymbol{r}' - \boldsymbol{r}|$. We assume that the strength $\mathcal{F}$ of the interparticle force is a smooth function of $r = |\boldsymbol{r}' - \boldsymbol{r}|$ that decays fast enough so that the integral $\int_0^\infty r^N \mathcal{F}(r) dr$ is convergent. Moreover, the short-range character of the interaction makes it reasonable to assume that the natural length scales of the system are large compared with the range of distances within which $\mathcal{F}(r)$ is substantially different from zero. In particular, over this range of distances the spatial density $\rho(\boldsymbol{r})$ can be approximated as $\rho(\boldsymbol{r}') = \rho(\boldsymbol{r}) + (\boldsymbol{r}' - \boldsymbol{r})(\nabla \rho)$ (see Ref. [8] for a full discussion). Within this approximation, the force $\boldsymbol{F}_{\text{int}}(\boldsymbol{r})$ acting on a particle located at $\boldsymbol{r}$, due to the interaction with the other particles, can be written as

$$\boldsymbol{F}_{\text{int}} = -\mathcal{G} \nabla \rho\,, \tag{7}$$

where $\mathcal{G} = \int r \mathcal{F}(r) d^N \boldsymbol{r} = \sigma_{N-1} \int_0^\infty r^N \mathcal{F}(r) dr$, and σ_{N-1} stands for the total hypersolid angle corresponding to an $(N-1)$-dimensional sphere (that is, to the hypersurface of an N-dimensional ball). One has $\sigma_0 = 2$, $\sigma_1 = 2\pi$, and $\sigma_2 = 4\pi$. In one dimension, the above force F_{int} formally corresponds to the one arising from the interaction potential between pairs of particles given by $\mathcal{V}(x_1, x_2) = \mathcal{G}\delta(x_2 - x_1)$, where $\delta(x)$ is Dirac's delta function and $x_{1,2}$ are the locations of the two particles.

The above explained components of the total force acting on a test particle imply that its motion is governed by

$$m\ddot{\boldsymbol{r}} = -\mathcal{G}\nabla\rho - \nabla W - \alpha\, \dot{\boldsymbol{r}} \left| \frac{\dot{\boldsymbol{r}}}{v_0} \right|^{\lambda}. \tag{8}$$

If the term $m\ddot{\boldsymbol{r}}$, corresponding to inertial effects, is much smaller than the other terms appearing in (8), then one has overdamped motion,

$$\dot{\boldsymbol{r}} \left| \frac{\dot{\boldsymbol{r}}}{v_0} \right|^{\lambda} = -\frac{\mathcal{G}}{\alpha}(\nabla\rho) - \frac{1}{\alpha}(\nabla W). \tag{9}$$

Solving (9) for $\dot{\boldsymbol{r}}$, one obtains

$$\dot{\boldsymbol{r}} = -\left[\frac{\mathcal{G}}{\alpha}(\nabla\rho) + \frac{1}{\alpha}(\nabla W)\right] \left| \frac{1}{v_0} \left[\frac{\mathcal{G}}{\alpha}(\nabla\rho) + \frac{1}{\alpha}(\nabla W)\right] \right|^{-\frac{\lambda}{1+\lambda}}. \tag{10}$$

The particle density $\rho(x,t)$, associated with a system of interacting particles whose motion is governed by (9), satisfies a continuity equation $\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0$, where the density current is

$$\boldsymbol{J} = -\rho \left[\frac{\mathcal{G}}{\alpha}(\nabla\rho) + \frac{1}{\alpha}(\nabla W)\right] \times \left| \frac{1}{v_0} \left[\frac{\mathcal{G}}{\alpha}(\nabla\rho) + \frac{1}{\alpha}(\nabla W)\right] \right|^{-\frac{\lambda}{1+\lambda}}. \tag{11}$$

After the identifications $D \to \frac{\mathcal{G}}{2\alpha}$ and $U \to \frac{W}{\alpha}$, this continuity equation reduces to

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left\{ \rho\, [\nabla(2D\rho + U)] \left| \frac{1}{v_0} \nabla(2D\rho + U) \right|^{-\frac{\lambda}{1+\lambda}} \right\}, \tag{12}$$

which constitutes a (highly) nonlinear Fokker-Planck equation. Equation (12) is the evolution equation governing the evolution of the density of a many-body system as the one considered in Ref. [8], when one has nonlinear, power-law drag forces. Indeed, when the drag forces are linear [corresponding the $\lambda = 0$ in the drag force (6)], the above equation coincides with the $q = 0$ case of the power-law NLFPE (5), which is relevant in connection with the thermostatistics of systems of vortices in type-II superconductors [8,10,15].

IV. STATIONARY SOLUTIONS AND H-THEOREM

In this section, we are going to investigate the stationary solutions of the evolution equation (12) and also obtain an H-theorem for this equation. When doing this, it will prove enlightening to consider a more general equation that admits (12) as a particular case. We shall then discuss the following NLFPE:

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left(\rho \left\{ \nabla \left[\mathcal{D}\left(\frac{2-q}{1-q}\right)\left(\frac{\rho}{\rho_0}\right)^{1-q} + U \right] \right\} \times \left| \frac{1}{v_0} \nabla \left[\mathcal{D}\left(\frac{2-q}{1-q}\right)\left(\frac{\rho}{\rho_0}\right)^{1-q} + U \right] \right|^{-\frac{\lambda}{1+\lambda}} \right). \tag{13}$$

As is the case for the power-law NLFPE (1), if one wants to work with dimensional quantities (as opposed to dimensionless ones), it is convenient to introduce the constant ρ_0 with dimensions of inverse volume. The NLFPE equation (12) obtained in the previous section constitutes the particular case of (13) corresponding to $q = 0$. Indeed, it can be verified that (13) reduces to (12) if one sets $q = 0$ and $D = \mathcal{D}/\rho_0$. As with the power-law NLFPE (1), we shall assume that $(2-q)\mathcal{D} > 0$. Equation (13) may look, at first sight, a bit intimidating. However, some physically relevant aspects of this equation lend themselves to analytical treatment. We are going to prove that (13) admits q-exponential stationary solutions and that it satisfies an H-theorem related to the S_q entropies.

A. Stationary solutions

The NLFPE (13) admits stationary solutions $\rho_q(\boldsymbol{r})$ that satisfy

$$\rho_q \nabla\left[\mathcal{D}\left(\frac{2-q}{1-q}\right)\left(\frac{\rho_q}{\rho_0}\right)^{1-q} + U\right] = 0, \qquad (14)$$

which implies that in the region where $\rho_q \neq 0$,

$$\mathcal{D}(2-q)\left(\frac{\rho_q}{\rho_0}\right)^{1-q} + (1-q)U = \eta, \qquad (15)$$

with η a constant. It follows from (15) that

$$\rho_q(\boldsymbol{r}) = \rho_0\left[\frac{\eta}{(2-q)\mathcal{D}}\right]^{\frac{1}{1-q}}\left[1-\frac{1-q}{\eta}U(\boldsymbol{r})\right]_+^{\frac{1}{1-q}}. \qquad (16)$$

After making the identifications

$$\beta = 1/\eta, \quad A = \left[\frac{\eta}{(2-q)\mathcal{D}}\right]^{\frac{1}{1-q}}, \qquad (17)$$

one finds that $(2-q)\mathcal{D}\beta = A^{q-1}$ and that the stationary solution (16) has the same form as the stationary solution (2) of the power-law NLFPE (1). This means that the stationary densities $\rho_q(\boldsymbol{r})$ of systems of confined particles interacting via short-range forces, and performing overdamped motion, do not depend on the exponent λ characterizing the nonlinear dependence of the drag forces on velocity. This is physically reasonable, because the stationary solutions of the NLFPE correspond to configurations of the particles for which the interaction forces between them are balanced with the forces due to the confining potential. However, the invariance of the stationary solutions has potentially interesting implications for the thesmostatistical theory associated with the S_q nonadditive entropies. Some important applications of this theory are based, essentially, on the fact that the NLFPEs describing the systems under consideration admit maximum S_q entropy stationary solutions. The fact that these solutions are robust, in the sense of being preserved when nonlinearities in the drag forces are at work, may considerably enlarge the range of physical scenarios where the thermostatistics based on the S_q measures is relevant.

B. *H*-Theorem

Let us consider the quantity

$$\mathcal{H} = \frac{\mathcal{D}}{1-q}\int \rho\left(\frac{\rho}{\rho_0}\right)^{1-q} d^N\boldsymbol{r} + \int \rho\, U\, d^N\boldsymbol{r}. \qquad (18)$$

One has that

$$\frac{d\mathcal{H}}{dt} = \mathcal{D}\left(\frac{2-q}{1-q}\right)\int\left(\frac{\rho}{\rho_0}\right)^{1-q}\frac{\partial\rho}{\partial t}\, d^N\boldsymbol{r} + \int U\frac{\partial\rho}{\partial t}\, d^N\boldsymbol{r}. \qquad (19)$$

We now substitute $\partial\rho/\partial t$ in (19) by the right-hand side of the NLFPE (13). After some computations that involve an integration by parts, one can verify that

$$\frac{d\mathcal{H}}{dt} = -\int \rho\left\{\mathcal{D}\left(\frac{2-q}{1-q}\right)\nabla\left[\left(\frac{\rho}{\rho_0}\right)^{1-q}\right] + \nabla U\right\}^2 \times\left|\frac{1}{v_0}\left[\mathcal{D}\left(\frac{2-q}{1-q}\right)\nabla\left[\left(\frac{\rho}{\rho_0}\right)^{1-q}\right] + \nabla U\right]\right|^{-\frac{\lambda}{1+\lambda}} d^N\boldsymbol{r} \leqslant 0, \qquad (20)$$

which constitutes an H-theorem for the evolution equation (13). The condition $d\mathcal{H}/dt = 0$ only holds for stationary solutions of the NLFPE (13) complying with (14). The functional $\mathcal{H}$ satisfying the H-theorem is closely related to the S_q entropies. In fact, the H-theorem (20) can be expressed in terms of a free-energy-like quantity, adopting the form (4), i.e.,

$$\frac{d}{dt}[\langle U\rangle - (\mathcal{D}/k)S_{q^*}[\rho]] \leqslant 0, \qquad (21)$$

with $q^* = 2-q$. We see here another interesting invariance. The form of free-energy-like quantity $\mathcal{H}$ complying with an H-theorem is preserved when one introduces power-law nonlinearities in the velocity dependence of the drag forces. On the contrary, the rate of change of this quantity, $d\mathcal{H}/dt$, does depend on the exponent λ characterizing the way in which drag depends on velocity.

V. AN EXAMPLE WITH STRETCHED *q*-EXPONENTIAL TIME-DEPENDENT SOLUTIONS

We are now going to discuss a particular, one-dimensional example of the NLFPE (12) that admits exact, time-dependent solutions having a stretched q-exponential form. Let us consider the NLFPE

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial x}\left\{\rho\left[\frac{\partial}{\partial x}(2D\rho + U)\right]\left|\frac{1}{v_0}\frac{\partial}{\partial x}(2D\rho + U)\right|^{-\frac{\lambda}{1+\lambda}}\right\}, \qquad (22)$$

which constitutes a one-dimensional instance of the NLFPE (12), corresponding to the $q = 0$ case of the NLFPE (13), with

$$U(x) = C\left|\frac{x}{x_0}\right|^{\delta} x^2, \qquad (23)$$

where C and x_0 are positive constants with dimensions of inverse time and length, respectively. In order for the potential $U(x)$ to be confining, and the NLFPE (22) to admit stationary solutions, one must have $\delta > -2$. The NLFPE (22) describes a set of particles interacting via short-range forces, confined by the potential $W = \alpha U$, and performing overdamped motion under the effect of the nonlinear drag force (6).

Replacing the ansatz

$$\rho(x,t) = A(t)\left[1-(1-q)\beta(t)\left|\frac{x}{x_0}\right|^{\delta} x^2\right]_+^{\frac{1}{1-q}}, \qquad (24)$$

into the evolution equation (22), it is possible to verify, after some algebra, that (24) constitutes, for $q = 0$, a solution of (22) provided that $\delta = \lambda$, and the time-dependent parameters A and β satisfy an appropriate pair of coupled equations of motion. Indeed, for the alluded q and δ one has

$$\frac{\partial}{\partial x}(2D\rho + U) = (\lambda+2)(C - 2DA\beta)\left|\frac{x}{x_0}\right|^{\lambda} x,$$

$$\left[\frac{\partial}{\partial x}(2D\rho + U)\right]\left|\frac{1}{v_0}\frac{\partial}{\partial x}(2D\rho + U)\right|^{-\frac{\lambda}{1+\lambda}}$$
$$= (\lambda+2)^{\frac{1}{1+\lambda}}(C - 2DA\beta)\left|\frac{x_0}{v_0}(2DA\beta - C)\right|^{-\frac{\lambda}{1+\lambda}} x. \quad (25)$$

Inserting the ansatz (24) into the evolution equation (22), and using the second equation in (25), it can be verified that (24) is a solution of (22) if A and β comply with the following coupled ordinary differential equations:

$$dA/dt = R\,A,$$
$$d\beta/dt = (\lambda+2)\,R\,\beta, \quad (26)$$

where the time-dependent quantity R is given by

$$R = (\lambda+2)^{\frac{1}{1+\lambda}}(C - 2DA\beta)\left|\frac{x_0}{v_0}(2DA\beta - C)\right|^{-\frac{\lambda}{1+\lambda}}. \quad (27)$$

Note that for $\lambda = 0$ one has $\delta = 0$, and the density (24) corresponds to a time-dependent q-Gaussian solution of a power-law NLFPE associated with linear drag and with a quadratic potential. We shall refer to the density (24) as a "stretched q-exponential." The solution (24) has cut-off points at $x = \pm x_m = \pm x_0(\beta x_0^2)^{-\frac{1}{\lambda+2}}$. The normalization of (24) is given by

$$N = \int_{-x_m}^{+x_m} \rho(x,t)dx = \left[\frac{2(\lambda+2)}{\lambda+3}\right] A x_0 \left(\beta x_0^2\right)^{-\frac{1}{\lambda+2}}. \quad (28)$$

It can be verified that the equations of motion (26) imply that $dN/dt = 0$ which, in turn, means that $A(x_0^2\beta)^{-\frac{1}{\lambda+2}}$ is an integral of motion of the system (26). In other words,

$$\frac{A(t)}{A_0} = \left[\frac{\beta(t)}{\beta_0}\right]^{\frac{1}{\lambda+2}}, \quad (29)$$

where $A_0 > 0$ and $\beta_0 > 0$ are, respectively, the initial values of the parameters A and β.

The equations of motion (26) guarantee that $A(t) > 0$ and $\beta(t) > 0$ for all times $t > 0$. It follows from (26) and (27) that the values $A_{\rm stat}$ and $\beta_{\rm stat}$ of the parameters (A, β), corresponding to the stationary solution, satisfy $2DA_{\rm stat}\beta_{\rm stat} = C$. Combining this relation with (29), one obtains

$$A_{\rm stat} = A_0\left[\frac{C}{2D\beta_0 A_0}\right]^{\frac{1}{3+\lambda}},$$
$$\beta_{\rm stat} = \frac{C}{2DA_0}\left[\frac{2D\beta_0 A_0}{C}\right]^{\frac{1}{3+\lambda}}. \quad (30)$$

The time evolution of the parameters $A(t)$ and $\beta(t)$ are illustrated in Figs. 1 and 2, respectively, for a system characterized by $x_0 = 1$, $v_0 = 1$, $D = 1$, $C = 1$, and $\lambda = 1$. The curves shown in Figs. 1 and 2 correspond to different initial values A_0 and β_0, all of them satisfying the same normalization $N = 1$. The initial values β_0 are indicated in the figures. The corresponding initial values A_0 are obtained from the normalization condition, setting at the initial time the right-hand side of (28) equal to 1. It is observed that, in all these cases,

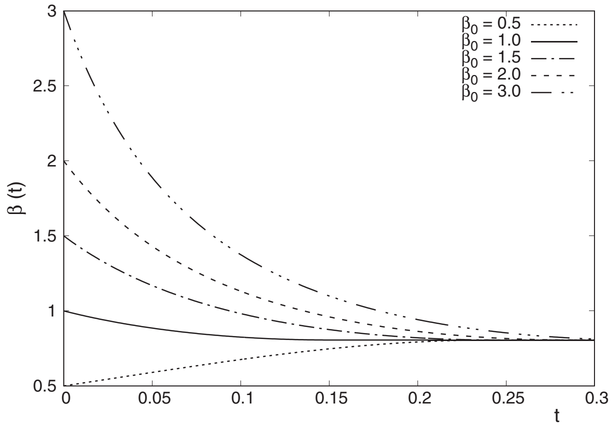

FIG. 1. Evolution of parameter β from the time-dependent solution (24) of the NLFPE (22) for different initial conditions A_0, β_0, and for $q = 0$, $x_0 = 1$, $v_0 = 1$, $D = 1$, $C = 1$, and $\lambda = 1$. The initial values of β are indicated in the figure. The corresponding initial values of A are determined from the normalization condition $N = 1$. Parameter β has dimensions of inverse squared length and is measured in units of $(1/x_0^2)$. Time t is measured in units of (x_0/v_0).

the solution of the NLFPE relaxes to the stationary solution, corresponding to the parameter values $A_{\rm stat}$ and $\beta_{\rm stat}$.

Trajectories in the (β, A)-parameter plane, corresponding to different initial conditions and different normalizations, are depicted in Fig. 3. The curve corresponding to the stationary values $\beta_{\rm stat}$, $A_{\rm stat}$ is also plotted. Each point on this curve represents a stationary solution with a different normalization N. A time-dependent solution (24) corresponds to a point that moves along one of the trajectories shown in Fig. 3. As $t \to \infty$, it approaches a stationary point $\beta_{\rm stat}$, $A_{\rm stat}$, either from the left or from the right, depending on whether the initial values of β and A are, respectively, smaller or larger than the corresponding stationary values $\beta_{\rm stat}$ and $A_{\rm stat}$.

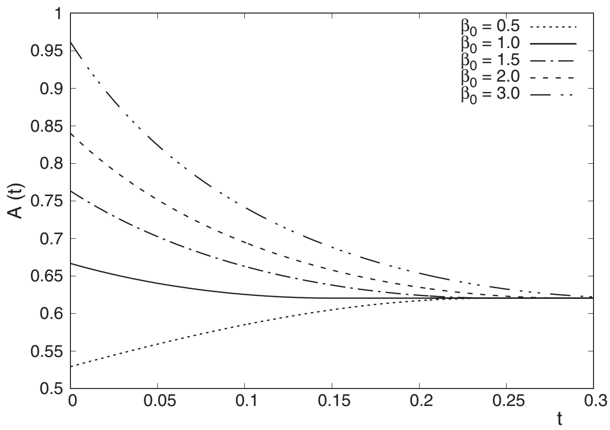

FIG. 2. Evolution of parameter A from the time-dependent solution (24) of the NLFPE (22) for $q = 0$, $x_0 = 1$, $v_0 = 1$, $D = 1$, $C = 1$, and $\lambda = 1$. The initial conditions A_0, β_0 considered in this figure are the same as in Fig. 1. Parameter A has dimensions of inverse length and is measured in units of $(1/x_0)$. Time t is measured in units of (x_0/v_0).

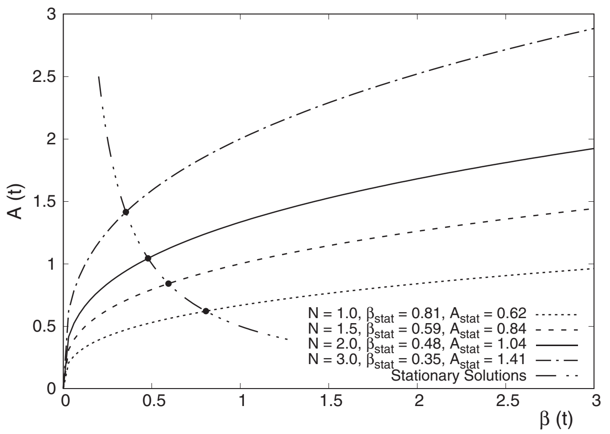

FIG. 3. Trajectories in the (β, A) plane corresponding to different initial conditions, plotted together with the hyperbola $C = 2DA_{\rm stat}\beta_{\rm stat}$, associated with the (β, A) values describing stationary solutions. The relevant parameters characterizing the system are the same as in Figs. 1 and 2. A is measured in units of $(1/x_0)$ and β in units of $(1/x_0^2)$.

We shall now determine explicitly, for the time-dependent solution (24) ($q = 0$), the time evolution of the quantity (18) that satisfies the H-theorem (20). If one replaces the ansatz (24) with $q = 0$, into (18), one obtains

$$\mathcal{H} = \frac{N}{2\lambda + 5}\left[2(\lambda+2)DA + \frac{C}{\beta}\right]. \tag{31}$$

The stationary value of $\mathcal{H}$ is

$$\mathcal{H}_{\rm stat} = \frac{\lambda+3}{2\lambda+5}\left(\frac{NC}{\beta_{\rm stat}}\right). \tag{32}$$

The evolution of the quantity $\mathcal{H}$, satisfying the H-theorem, is exhibited in Fig. 4 for different initial conditions and for the same set of parameters as those corresponding to Figs. 1 and 2. It can be appreciated in this figure that $\mathcal{H}$ decreases monotonically with time, consistently with the H-theorem and in all cases tends to a value corresponding to the stationary solution.

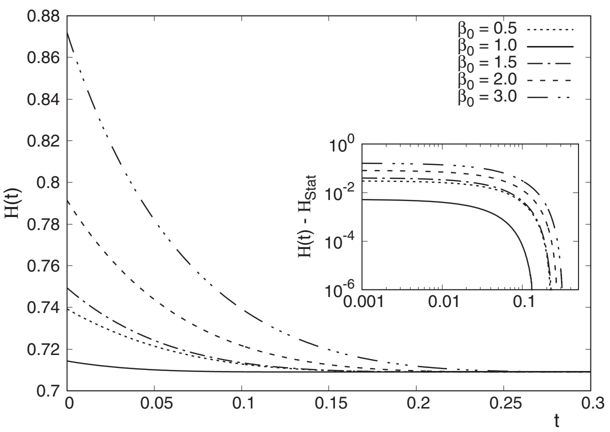

FIG. 4. Evolution of the quantity $\mathcal{H}$ satisfying the H-theorem. The inset depicts, in log-log scale, the behavior of $\mathcal{H} - \mathcal{H}_{\rm stat}$ against time. The relevant parameters characterizing the system, and the initial conditions, are the same as in Figs. 1 and 2. $\mathcal{H}$ is measured in units of $(v_0 x_0)$ and t in units of (x_0/v_0).

Let us consider the special case of no confinement ($C = 0$). In this case, the system of particles spreads due to the repulsive interactions, and the evolution of the particle density ρ is determined by

$$dA/dt = R^* A, \quad d\beta/dt = (\lambda+2)\, R^*\, \beta, \tag{33}$$

where [note that the quantity inside the absolute value symbols in (27) is now positive]

$$R^* = -\left[2(\lambda+2)D\left(\frac{x_0}{v_0}\right)A\beta\right]^{\frac{1}{1+\lambda}}\left(\frac{v_0}{x_0}\right). \tag{34}$$

Using (29), it follows from (33) that

$$\frac{dA}{dt} = -\left(\frac{v_0}{x_0}\right)\left[2(\lambda+2)D\left(\frac{x_0}{v_0}\right)\beta_0 A\left(\frac{A}{A_0}\right)^{\lambda+2}\right]^{\frac{1}{1+\lambda}} A. \tag{35}$$

The above equation has solution

$$A(t) = A_0\left[1 + \frac{t}{t_0}\right]^{\Gamma}, \tag{36}$$

where

$$\Gamma = -\left(\frac{1+\lambda}{3+\lambda}\right), \quad t_0 = \left(\frac{1+\lambda}{3+\lambda}\right)\left(\frac{x_0}{v_0}\right)\left[2(\lambda+2)D\left(\frac{x_0}{v_0}\right)\beta_0 A_0\right]^{-\frac{1}{1+\lambda}}. \tag{37}$$

The time-dependent parameter β is then given by

$$\beta(t) = \beta_0\left[1 + \frac{t}{t_0}\right]^{(2+\lambda)\Gamma}. \tag{38}$$

Note that, as the time-dependent density relaxes to the stationary one, both A and β decay in a q-exponential fashion, with λ-dependent effective values of the parameter q, respectively, given by

$$q_A^{(\rm relax)} = 1 - \frac{1}{\Gamma} = \frac{2(2+\lambda)}{1+\lambda}, \quad q_\beta^{(\rm relax)} = 1 - \frac{1}{(2+\lambda)\Gamma} = \frac{\lambda^2+4\lambda+5}{(1+\lambda)(2+\lambda)}. \tag{39}$$

In the present system we see that more than one q index emerges. In fact, generally speaking, three or more such indices do appear in many applications of q statistics. These sets of q values are usually referred to as the q triplet and its extensions [43–45].

When there is no confinement ($C = 0$) the quantity satisfying the H-theorem evolves according to

$$\mathcal{H} = NDA_0\left[\frac{2(\lambda+2)}{2\lambda+5}\right]\left[1 + \frac{t}{t_0}\right]^{\Gamma}, \tag{40}$$

meaning that this free-energy-like measure also decays as a q exponential, with $q_{\mathcal{H}}^{(\rm relax)} = q_A^{(\rm relax)}$ [see the first line of

Eq. (39)]. This implies that, asymptotically, $\mathcal{H}$ decays as a power law with exponent Γ.

VI. EXACT SOLUTIONS FOR GENERAL q-VALUES

The nonlinear Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left(\rho\left\{\frac{\partial}{\partial x}\left[\mathcal{D}\left(\frac{2-q}{1-q}\right)\left(\frac{\rho}{\rho_0}\right)^{1-q} + U\right]\right\}\right.$$
$$\left.\times\left|\frac{1}{v_0}\frac{\partial}{\partial x}\left[\mathcal{D}\left(\frac{2-q}{1-q}\right)\left(\frac{\rho}{\rho_0}\right)^{1-q} + U\right]\right|^{-\frac{\lambda}{1+\lambda}}\right), \quad (41)$$

with $(2-q)\mathcal{D} > 0$ and a potential U given by (23), also admits time-dependent solutions of the form (24) for general q values, provided that $\delta = \lambda$ and, in order for the solution to be normalizable, $q < \lambda + 3$ (note that these restrictions allow for both $q < 1$ and $q > 1$). The time-dependent parameters $A(t)$ and $\beta(t)$ must obey the couple ordinary differential equations,

$$\frac{dA}{dt} = R_q A,$$
$$\frac{d\beta}{dt} = (\lambda+2)R_q\beta, \quad (42)$$

with

$$R_q = (\lambda+2)^{-\frac{\lambda}{1+\lambda}}\left[C - (2-q)\mathcal{D}\beta\left(\frac{A}{\rho_0}\right)^{1-q}\right]$$
$$\times\left|x_0\left[C - (2-q)\mathcal{D}\beta\left(\frac{A}{\rho_0}\right)^{1-q}\right]\right|^{-\frac{\lambda}{1+\lambda}}. \quad (43)$$

The evolution equations (43) imply that $A(t)/A_0 = (\beta(t)/\beta_0)^{1/(2+\lambda)}$ which, in turn, guarantees that the normalization N of the time-dependent density is constant in time [see Eq. (28)].

The stationary solution of (42) is

$$A_{\text{stat}} = A_0\left(\frac{\rho_0}{A_0}\right)^{\frac{1}{1+(1-q)(2+\lambda)}}\left[\frac{C}{(2-q)\mathcal{D}\beta_0}\right]^{\frac{1-q}{1+(1-q)(2+\lambda)}}$$
$$\beta_{\text{stat}} = \beta_0\left(\frac{\rho_0}{A_0}\right)^{\frac{2+\lambda}{1+(1-q)(2+\lambda)}}\left[\frac{C}{(2-q)\mathcal{D}\beta_0}\right]^{\frac{(1-q)(2+\lambda)}{1+(1-q)(2+\lambda)}}. \quad (44)$$

The evolution equation (41) is the one-dimensional instance of the nonlinear Fokker-Planck equation (13). For $q = 0$ equation (41) reduces to (22) if one sets $D = \mathcal{D}/\rho_0$.

It is worth mentioning that probability density functions of the q-exponential form having [as the stretched q-exponential solutions (24) with $\delta > 0$ have] arguments neither linear nor quadratic in the relevant dynamical variables are actually observed experimentally. For instance, densities of that kind describe experimental data on the distribution of velocity differences in turbulent Couette-Taylor flows [46]. In a different context, stretched q-exponential functions are also relevant in connection with spin-glass relaxation in neutron spin-echo experiments [47].

VII. CONCLUSIONS

In the present work, we have considered many-body systems consisting of confined particles interacting via short-range forces, while performing overdamped motion, under the effect of nonlinear drag forces with a power-law dependence on velocity. In order to explore the thermostatistics of this kind of systems, we derived a family of nonlinear Fokker-Planck equations that provide an effective description of the concomitant dynamics. We investigated the main properties of these nonlinear Fokker-Planck equations. In particular, we proved that they have maximum S_q entropy stationary solutions and that they admit a free-energy-like quantity satisfying an H-theorem. This quantity involves a power-law nonadditive entropic functional S_q characterized by an appropriate value of the q parameter. We also obtained, for particular confining potentials and for the limit case of no confinement, semianalytic time-dependent solutions of the above-mentioned nonlinear Fokker-Planck equation, exhibiting the form of stretched q-exponential densities. An interesting aspect of these solutions is that they include instances corresponding to $q > 1$, where the q-exponential densities have power-law asymptotically decaying long tails. This is in contrast with various recent studies on the S_q-NLFPE connection, which focused on $q < 1$ solutions with compact support (see Ref. [36] and references therein).

The present results imply that the strong connection that exists among the aforementioned many-body problems, their associated nonlinear Fokker-Planck equations, and the thermostatistics based on the S_q nonadditive entropies is preserved when power-law nonlinearities in the drag forces are introduced.

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