

Consistent thermodynamic framework for interacting particles by neglecting thermal noiseFernando D. Nobre,^{1,4,*} Evaldo M. F. Curado,^{1,4,†} Andre M. C. Souza,^{2,4,‡} and Roberto F. S. Andrade^{3,4,§}¹*Centro Brasileiro de Pesquisas Físicas, Rua Xavier Sigaud 150, 22290-180 Rio de Janeiro-RJ, Brazil*²*Departamento de Física, Universidade Federal de Sergipe, 49100-000 São Cristóvão-SE, Brazil*³*Instituto de Física, Universidade Federal da Bahia, 40210-340 Salvador-BA, Brazil*⁴*National Institute of Science and Technology for Complex Systems, Rua Xavier Sigaud, 150 22290-180 Rio de Janeiro-RJ, Brazil*

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An effective temperature θ , conjugated to a generalized entropy s_q , was introduced recently for a system of interacting particles. Since θ presents values much higher than those of typical room temperatures $T \ll \theta$, the thermal noise can be neglected ($T/\theta \simeq 0$) in these systems. Moreover, the consistency of this definition, as well as of a form analogous to the first law of thermodynamics, $du = \theta ds_q + \delta W$, were verified lately by means of a Carnot cycle, whose efficiency was shown to present the usual form, $\eta = 1 - (\theta_2/\theta_1)$. Herein we explore further the heat contribution $\delta Q = \theta ds_q$ by proposing a way for a heat exchange between two such systems, as well as its associated thermal equilibrium. As a consequence, the zeroth principle is also established. Moreover, we consolidate the first-law proposal by following the usual procedure for obtaining different potentials, i.e., applying Legendre transformations for distinct pairs of independent variables. From these potentials we derive the equation of state, Maxwell relations, and define response functions. All results presented are shown to be consistent with those of standard thermodynamics for $T > 0$.

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I. INTRODUCTION

The macroscopic description of physical systems in equilibrium at a temperature T is done appropriately by thermodynamics in a very elegant manner [1–3]. In each physical situation, the identification of the adequate set of variables, and associated thermodynamic potentials, represents an important step for a suitable analysis. Studies are carried for $T > 0$, from which one can obtain low-temperature results and, subsequently, physical properties are derived for $T = 0$ by considering the limit $T \rightarrow 0$. In this limit, the Boltzmann-Gibbs (BG) statistical mechanics imposes that certain thermodynamic properties should present well-defined behavior, obeying the third law of thermodynamics, which implies that the entropy per particle, $s_{BG} \rightarrow 0$.

The concept of effective temperature has been used in the literature in many situations where $T > 0$ and, most curiously, also for $T = 0$. As examples of the former case, one could mention the Einstein (T_E) and Debye (T_D) temperatures in their respective crystalline-solid models, so distinct physical behaviors are obtained for low temperatures ($T \ll T_E, T_D$) and high temperatures ($T \gg T_E, T_D$). However, the Fermi temperature (T_F) in a Fermi-Dirac ideal gas is defined for $T = 0$, being directly related to the concentration of electrons, and so it may present large variations depending on the physical system. One should call the attention to the fact that previous effective-temperature concepts in the literature, present, to our knowledge, well-defined (i.e., fixed) values for each physical system. In contrast to these, in the present work we will deal with an effective temperature θ , to be defined later, that can be varied experimentally.

The H-theorem represents one of the most important results in standard nonequilibrium statistical mechanics [1,3], guaranteeing the approach to an equilibrium state. Essentially, this theorem expresses a well-defined sign for the time derivative of the entropy, i.e., $(ds_{BG}/dt) \geq 0$, in the case of an isolated system, or for the free energy $(df_{BG}/dt) \leq 0$ ($f_{BG} = u - Ts_{BG}$), in the case of a system in contact with a heat reservoir. A possible proof of this theorem may be achieved by considering the functional form of $s_{BG}[P(x,t)]$ in terms of the probability density $P(x,t)$ for finding a particle at a position between x and $x + dx$ in time t and making use of the linear Fokker-Planck equation for the time derivative $\partial P(x,t)/\partial t$. In this way, the H-theorem provides a direct connection between the linear Fokker-Planck equation and the BG entropy.

Following a similar procedure, proofs of the H-theorem have also been achieved recently by considering nonlinear Fokker-Planck equations (NLFPEs). Therefore, in the same way that the linear Fokker-Planck equation is associated to normal diffusion and to the BG entropy, the NLFPEs are usually related to anomalous-diffusion phenomena [4] and to generalized entropies (see, e.g., Refs. [5–7] among others). The NLFPEs have been widely investigated in the literature [8], motivated by an appropriate description of many complex physical systems. A particular interest has been dedicated to the NLFPE proposed in Refs. [9], related to Tsallis nonextensive statistical mechanics [10–13]. In particular, the q -Gaussian distribution, which represents a generalization of the standard Gaussian (recovered in the particular case $q = 1$), appears naturally from an extremization procedure of the entropy [13] or from the solution of the corresponding nonlinear Fokker-Planck equation [9]. This distribution has been very useful for experiments in many real systems [10–12]; among many, one could highlight: (i) the velocities of cold atoms in dissipative optical lattices [14,15]; (ii) the velocities of particles in quasi-two-dimensional dusty plasma [16]; (iii) the relaxation curves of RKKY spin glasses, like CuMn and AuFe [17]; (iv) single ions in radio-frequency traps interacting

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with a classical buffer gas [18]; (v) transverse momenta distributions at LEP [19] and LHC experiments [20]; and (vi) the overdamped motion of interacting vortices in type II superconductors [21–26].

Recently, the effective temperature θ was introduced within the context of the last application above, namely interacting vortices in type II superconductors [25]. The quantity θ was shown to represent an appropriate definition of effective temperature for this system, exhibiting properties very similar to those of the usual thermodynamic temperature T , being [25,26]: (a) a positive quantity by definition and (b) thermodynamically conjugated to a generalized entropy per particle, s_q , with $q = 2$ (to be called hereafter s_2), characteristic of nonextensive statistical mechanics (in this way, a heat contribution was defined, $\delta Q = \theta ds_2$); (c) proportional to the density of vortices n (this property yields the desirable possibility for varying θ , since recent experimental researches in type II superconductors has led to considerable advances in the ability of controlling many properties of these vortices, including their density [27–30]); (d) characterized by values that are much higher than typical room temperatures ($\theta \gg T$), so the thermal noise can be neglected as a good approximation ($T/\theta \simeq 0$); (e) physically interpreted in terms of the variance of the vortex positions, $\theta \propto \langle x^2 \rangle^{3/2}$ [26]; and (f) consistent with the definition of a Carnot cycle, whose efficiency was shown to be $\eta = 1 - (\theta_2/\theta_1)$, where θ_1 and θ_2 represent the effective temperatures associated with the isothermal transformations of the cycle, with $\theta_1 > \theta_2$ [26]. In order to achieve this latter result, an infinitesimal-work term δW was introduced, leading to a proposal for the first law of thermodynamics.

In the present work we reinforce the adequacy of the effective temperature θ , as well as of first-law form, introduced in Refs. [25] and [26], respectively, by showing the consistency of an associated thermodynamic framework. In the next section we define the physical system under consideration and review some results of previous works [21–26]. In Sec. III we explore the heat contribution, $\delta Q = \theta ds_2$, by considering systems in thermal contact in such a way to exchange heat among themselves. Important concepts like thermal equilibrium and heat reservoir are introduced, and the zeroth principle is established. In Sec. IV we define physical transformations and study the Carnot refrigerator, as well as an additional cycle, namely the Otto cycle. It is shown that the Carnot cycle (as well as the Carnot refrigerator) are special for this system, in the sense that its efficiency keeps its usual form, contrary to the Otto cycle, whose efficiency presents a form that differs from the one found in standard thermodynamics. In Sec. V we explore further the first law in its infinitesimal form by applying Legendre transformations for distinct pairs of independent variables in order to derive different potentials and Maxwell relations. In Sec. VI we introduce some response functions, showing that they present properties very similar to those of standard thermodynamics. Finally, in Sec. VII we present our conclusions.

II. AN OVERVIEW OF PREVIOUS RESULTS

In this section we present a brief review of previous results on the present problem, discussing the connection of a NLFPE with the system of interacting type II superconducting vortices

[21–24], the concept of an effective temperature and the reason for neglecting thermal noise in this problem [25], a form analogous to the first law [26], and some results very similar to those of standard thermodynamics [25,26].

A. Physical system and effective-temperature concept

The physical system considered herein concerns interacting vortices under overdamped motion; it will be defined below and it has been used in the literature to model flux lines, and their associated vortices, in disordered type II superconductors (see, e.g., Refs. [27–33]).

The present study is motivated by a good agreement found between the vortex-position stationary-state [21] (as well as time-dependent [22,23]) distributions obtained by means of molecular-dynamics simulations and the analytical solution of the associated NLFPE calculated in Refs. [9].

The equation of motion of a flux line i under overdamped motion [i.e., with $(dv_i/dt) = 0$], in a medium with an effective friction coefficient μ , is given by

$$\mu \mathbf{v}_i = \mathbf{F}_i^{\text{pp}} + \mathbf{F}_i^{\text{ext}} \quad (i = 1, 2, \dots, N). \quad (2.1)$$

In this equation, \mathbf{v}_i represents the velocity, whereas the terms on the right-hand side depict the forces acting on flux line i . The particle-particle interactions \mathbf{F}_i^{pp} denote contributions of the other vortices [each vortex interacts with the remaining $(N - 1)$ vortices], whereas $\mathbf{F}_i^{\text{ext}}$ represents an external force acting on vortex i . The vortex-vortex interactions are repulsive and radially symmetric and are given by [31–33]

$$\mathbf{F}_i^{\text{pp}} = \frac{f_0}{2} \sum_{j \neq i} K_1(r_{ij}/\lambda) \hat{\mathbf{r}}_{ij}, \quad (2.2)$$

where $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ stands for the distance between vortices i and j and $\hat{\mathbf{r}}_{ij} = (\mathbf{r}_i - \mathbf{r}_j)/r_{ij}$ is a vector defined along the axis joining them. Moreover, K_1 represents a modified Bessel function of the second kind of order 1 and f_0 is a positive constant. These interactions are defined in terms of a characteristic length scale λ , known as the London penetration length; other linear measures of this system are expressed in units of λ . The external forces $\mathbf{F}_i^{\text{ext}}$ should be associated to a confining type of potential, so the system can reach a stationary state after a sufficiently long time. Herein this problem will be considered in a two-dimensional box of dimensions L_x and L_y to conform with previous numerical simulations [21–23], although generalizations to higher dimensions are straightforward.

For an external force in the x direction, $\mathbf{F}^{\text{ext}} = -A(x)\hat{\mathbf{x}}$, a coarse-graining procedure in Eq. (2.1) leads to the following NLFPE [21,22,33]:

$$\begin{aligned} \mu \frac{\partial P(x,t)}{\partial t} = & - \frac{\partial [A(x)P(x,t)]}{\partial x} + 2D \frac{\partial}{\partial x} \left\{ [\lambda P(x,t)] \frac{\partial P(x,t)}{\partial x} \right\} \\ & + kT \frac{\partial^2 P(x,t)}{\partial x^2}, \end{aligned} \quad (2.3)$$

where $D = N\pi f_0 \lambda^2 / L_y$. One should notice that the equation above also takes into account the effects of a heat bath at a temperature T (with k standing for the Boltzmann constant), whose contribution may be obtained in the standard way through the introduction of a thermal noise in the system [3].

The most important aspect of Eq. (2.3) concerns the presence of two competing diffusion terms: apart from the usual linear diffusion contribution, one finds also a nonlinear diffusion term. The latter contribution, essentially generated from the particle-particle interactions within the coarse-graining procedure, plays a crucial role for an appropriate description of the low-temperature behavior of this system and especially of its superconducting state.

It appears intuitive from Eq. (2.3) to introduce an effective temperature θ , a concept that is supported by means of an H-theorem (to be discussed later),

$$k\theta \equiv D = \frac{N\pi f_0 \lambda^2}{L_y} = n\pi f_0 \lambda^2. \quad (2.4)$$

The quantity $k\theta$ presents the dimension of energy and is directly related to the density $n = N/L_y$, as well as to the interactions between vortices, being always positive. According to recent advances in experimental techniques, the density of vortices became a controllable quantity [27–30], leading to the desirable possibility of varying θ .

From the NLFPE of Eq. (2.3) one expects the following qualitative behaviors, depending on the parameters θ and T : (i) For $\theta \gg T$ the nonlinear-diffusion contribution prevails (in this regime the appropriate solution is given by a q -Gaussian distribution, typical of nonextensive statistical mechanics [21–26]); (ii) for $\theta \ll T$ one should recover the linear diffusion, associated with BG statistical mechanics [21,24]; and (iii) for $\theta \approx T$ one has an intermediate physical situation, interpolating between the two previous regimes. Although a time-dependent solution has not been found yet in this latter case, in Ref. [21] it was shown that the stationary-state solution of Eq. (2.3) is expressed in terms of a W Lambert function $W(z)$, defined implicitly through the equation $W(z) \exp[W(z)] = z$, where the ratio (θ/T) appears naturally in the variable z . According to the recent investigation of Ref. [25], values of θ in real systems are very high (e.g., in the compounds Pb-Bi, Nb-N, and $\text{YBa}_2\text{Cu}_3\text{O}_7$, one finds typically $\theta > 10^8$ K). Consequently, the appropriate physical situation for describing type II superconductors (where maximum critical temperatures are typically $T \approx 150$ K [31]) corresponds to regime (i) above (i.e., $\theta \gg T$), for which the nonlinear-diffusion contribution of Eq. (2.3) dominates completely, as expected. This is the reason for neglecting the linear-diffusion contribution in Refs. [22–26], as well as in the present study.

B. The NLFPE and associated entropy resulting from neglecting thermal noise

Based on the above arguments, the thermal noise can be neglected as a good approximation ($T/\theta \simeq 0$), so from now on we will be concerned with the following NLFPE:

$$\mu \frac{\partial P(x,t)}{\partial t} = -\frac{\partial [A(x)P(x,t)]}{\partial x} + 2D \frac{\partial}{\partial x} \left\{ [\lambda P(x,t)] \frac{\partial P(x,t)}{\partial x} \right\}. \quad (2.5)$$

It should be mentioned that this equation represents a particular case of the NLFPE usually considered in nonextensive statistical mechanics. The equation in Refs. [9] contains a more

general diffusion term, defined by replacing $2D[\lambda P(x,t)] \rightarrow \nu D[\lambda P(x,t)]^{\nu-1}$, where ν is a real number associated with the index q through $\nu = 2 - q$; Eq. (2.5) corresponds to the particular case $\nu = 2$ (leading to a q -Gaussian with $q = 0$ as its solution). The time-dependent solution of the general NLFPE was found for an initial condition $P(x,0) = \delta(x)$ and a harmonic external force, $A(x) = -\alpha x$ ($\alpha > 0$) [9],

$$P(x,t) = B(t)[1 + b(t)(1 - \nu)x^2]_+^{1/(\nu-1)}, \quad (2.6)$$

where $[y]_+ = y$, for $y > 0$, zero otherwise, and the time-dependent coefficients $B(t)$ and $b(t)$ are related to each other in order to preserve the normalization of $P(x,t)$ for all times t . The particular case of interest herein corresponds to $\nu = 2$, being identified with a q -Gaussian distribution of nonextensive statistical mechanics for $q = 0$ [10–12],

$$P(x,t) = B(t)[1 - b(t)x^2]_+, \quad (2.7)$$

and presenting a compact support in the interval $[-\bar{x}(t), \bar{x}(t)]$, where $\bar{x}(t) = b^{-1/2}(t)$.

One may prove the H-theorem using Eq. (2.5) and imposing a well-defined sign for the time derivative of the one-vortex free-energy functional [21–23],

$$f = u - \theta s_2; \quad u = \int_{-\bar{x}(t)}^{\bar{x}(t)} dx \phi(x) P(x,t), \quad (2.8)$$

where θ has to be identified precisely as the effective temperature of Eq. (2.4), and $\phi(x)$ represents the external potential associated with the force $A(x)$ in Eq. (2.5) [$A(x) = -d\phi(x)/dx$]. As usual, the internal energy is defined solely in terms of the external potential; the contribution of the interactions among vortices appears through the parameter θ . In order to satisfy the H-theorem, the associated entropy should be given by [21–23]

$$s_2[P] = k \left\{ 1 - \lambda \int_{-\bar{x}(t)}^{\bar{x}(t)} dx [P(x,t)]^2 \right\}. \quad (2.9)$$

It is important to remind that the distribution in Eq. (2.7) coincides with the one obtained through an extremization of the above entropy, considering the constraint for probability normalization, in addition to the constraint of Eq. (2.8) for the internal energy. Within the context of nonextensive statistical mechanics, by extremizing an entropy $s_\nu[P]$ considering a linear constraint for the internal energy [like the one in Eq. (2.8)], one obtains an equilibrium distribution given in terms of a q exponential with $q = 2 - \nu$ [10]. However, if one extremizes $s_\nu[P]$ with an escort constraint [34,35], one finds a distribution given in terms of a q exponential with $q = \nu$. Consequently, in the present work the linear constraint in Eq. (2.8) yields the “duality” $q \leftrightarrow (2 - q)$ between the distribution index $q = 0$ and the entropic exponent that appears in Eq. (2.9).

Two important points concerning the present approach should be stressed, as described below. (i) Similarly to the linear Fokker-Planck equation [3], the first term on the right-hand side of Eq. (2.5) carries the contribution of the external potential, whereas the second one comes from the interaction of a single flux line with the medium, and as mentioned, this latter term represents the effect of $N - 1$ vortices on the tagged vortex. Therefore, the distribution $P(x,t)$ will refer to one flux line of the above-mentioned system and so all physical quantities to be derived from this distribution (e.g.,

f , u , and s_2) will correspond to one-vortex properties. (ii) Based on the fact that $T/\theta \simeq 0$, the effects of thermal noise can be neglected as a good approximation. Consequently, certain thermodynamic properties, like entropy and specific heat, become negligible within BG statistical mechanics; however, a curious situation concerning the third law of thermodynamics was verified in Ref. [21], where $s_{BG} \rightarrow 0$, keeping the generalized entropic form $s_2 > 0$, as $T \rightarrow 0$. Hence, as shown in Refs. [25,26], the effective temperature θ appears as the variable thermodynamically conjugated to the generalized entropy s_2 .

C. The stationary state and some associated properties

One notices two competing terms on the right-hand side of Eq. (2.5), namely the confining external potential term and the nonlinear diffusion contribution produced by the repulsive interactions among the particles. Consequently, after a sufficiently long time, the system will attain a stationary state, where the particles appear at fixed positions, as shown in Refs. [21–23]. Such a stationary state corresponds to the one for which all thermodynamic-like properties will be calculated; for convenience, these quantities will be referred to by using the same nomenclature of standard thermodynamics.

The stationary-state solution of Eq. (2.5) is given by [21,22],

$$P_{st}(x) = \frac{\alpha}{4k\theta\lambda}(x_e^2 - x^2) = \frac{\alpha\lambda}{4k\theta} \left[\left(\frac{x_e}{\lambda} \right)^2 - \left(\frac{x}{\lambda} \right)^2 \right], \quad (2.10)$$

with $|x| < x_e$, where $x_e = (3k\theta\lambda/\alpha)^{1/3}$ is found from the normalization condition for $P_{st}(x)$. The above distribution presents a variance,

$$\langle x^2 \rangle = \int_{-x_e}^{x_e} dx x^2 P_{st}(x) = \frac{3^{2/3}}{5} \left(\frac{\lambda}{\alpha} \right)^{2/3} (k\theta)^{2/3} = \frac{x_e^2}{5}, \quad (2.11)$$

which yields

$$k\theta = \frac{5^{3/2}}{3} \frac{\alpha}{\lambda} \langle x^2 \rangle^{3/2}. \quad (2.12)$$

Therefore, the effective temperature θ is herein related to the particle-position variance, such that $\theta \propto \langle x^2 \rangle^{3/2}$, while in the classical dilute gas the temperature is related to the second moment of the corresponding velocity probability distribution, i.e., $T \propto \langle v^2 \rangle$.

Considering the distribution $P_{st}(x)$ in Eq. (2.10) one may calculate physical quantities in the stationary state, like the entropy and internal energy. Hence, Eqs. (2.8) and (2.9) lead, respectively, to

$$u = \int_{-x_e}^{x_e} dx \frac{\alpha x^2}{2} P_{st}(x) = \frac{3^{2/3}}{10} (\alpha\lambda^2)^{1/3} (k\theta)^{2/3}, \quad (2.13)$$

$$\frac{s_2}{k} = 1 - \lambda \int_{-x_e}^{x_e} dx [P_{st}(x)]^2 = 1 - \frac{3^{2/3}}{5} \left(\frac{\alpha\lambda^2}{k\theta} \right)^{1/3}. \quad (2.14)$$

Manipulating the equations above one may write the entropy in terms of the internal energy,

$$s_2(u, \alpha) = k \left[1 - \frac{3}{5} \left(\frac{\alpha\lambda^2}{10u} \right)^{1/2} \right], \quad (2.15)$$

where, following Ref. [26], we have written explicitly a dependence on the parameter α , i.e., $s_2 = s_2(u, \alpha)$, to be explored later. From Eq. (2.15) one obtains the fundamental relation,

$$\left(\frac{\partial s_2}{\partial u} \right)_\alpha = \frac{1}{\theta}, \quad (2.16)$$

which shows the appropriateness of the effective-temperature concept introduced in Ref. [25]. Moreover, this result suggests the definition of a type of energy exchange, $\delta Q = \theta ds_2$, hereafter to be called “heat exchange.”

D. The first law and equation of state

In Ref. [26] we have proposed a work contribution related to the external potential, $\delta W = \sigma d\alpha$, where the parameter α may be some controllable external field associated with work. This choice is supported by the fact that our distribution of particles is defined in the interval $[-x_e, x_e]$ with $x_e = (3k\theta\lambda/\alpha)^{1/3}$, so for a fixed θ , the volume occupied by the particles, $2x_e L_y$, decreases for increasing α . Furthermore, σ corresponds to the parameter conjugated to α (presenting dimensions $[\sigma] = L^2$) to be determined later. This proposal for an infinitesimal work δW , together the definition of an infinitesimal heat δQ given above, yields an equivalent to the first law [26],

$$du = \delta Q + \delta W = \theta ds_2 + \sigma d\alpha, \quad (2.17)$$

where δW corresponds to the work done by the external field on the system, and the dependence $s_2 = s_2(u, \alpha)$, introduced in Eq. (2.15), becomes clear now in Eq. (2.17). The consistency of this proposal for the first law will be shown throughout the rest of this paper. Besides the fundamental relation of Eq. (2.16), the first law proposed in Eq. (2.17) yields

$$\left(\frac{\partial s_2}{\partial \alpha} \right)_u = -\frac{\sigma}{\theta}. \quad (2.18)$$

Moreover, deriving Eq. (2.15) with respect to α , using Eq. (2.13), and equating the result with Eq. (2.18), one obtains the following equation of state:

$$\sigma = \frac{3^{2/3}}{10} \lambda^2 \left(\frac{k\theta}{\alpha\lambda^2} \right)^{2/3}. \quad (2.19)$$

Comparing the above expression with Eq. (2.13), one notices that $u = \sigma\alpha$, which represents a peculiarity of the present system. Particularly, it leads to a trivial enthalpy, $h = u - \sigma\alpha = 0$; this uncommon situation will be discussed in more detail later. The relation $u = \sigma\alpha$, involving the two conjugated parameters associated with the infinitesimal amount of work δW and the internal energy u , may be compared with the one for an ideal gas, namely $pv = 2u/3$ (valid for the classic case, as well as in both quantum statistics [3]).

Motivated by these results, the behavior of some stationary-state properties, like entropy and internal energy, were investigated for varying θ [25]; moreover, by defining appropriately physical transformations, a Carnot cycle was constructed in Ref. [26]. These investigations have shown that θ plays a role in the present problem very similar to the one of absolute temperature in standard thermodynamics. Following Refs. [25,26], herein we will analyze further properties related

to the entropic form s_2 of Eq. (2.9) and to the effective temperature of Eq. (2.4), showing that they present a nontrivial behavior, when neglecting thermal noise.

In the next section we will discuss novel aspects related to the contact between two systems at initially different values of the parameter θ ; based on this, we will show the validity of the zeroth law within the present framework.

III. SYSTEMS IN THERMAL CONTACT AND ZEROTH LAW

The system investigated herein, namely interacting vortices under overdamped motion, was studied numerically through molecular-dynamics simulations in Refs. [21–23]. These simulations considered N interacting vortices in a two-dimensional box of sizes L_x and L_y by applying a confining harmonic potential in the x direction, such as to guarantee a stationary state after a sufficiently long time. The remarkable agreement between the simulation data and the analytical solution of Eq. (2.5), for both time-dependent and stationary states, motivated us to carry on the study of further properties of this system, like those described above. Particularly, considering the stationary state described by the probability distribution of Eq. (2.10), a framework very similar to the one of standard thermodynamics was initiated in Refs. [25,26]; this analysis is continued and such similarity is reinforced herein.

In this section we discuss fundamental questions related to the energy transfer associated with the heat contribution $\delta Q = \theta ds_2$, which occurs by considering two of the above-mentioned systems in contact (to be referred to as systems 1 and 2, respectively). It is important to mention that since we are neglecting thermal effects, the present quantity δQ rather differs from the corresponding one of standard thermodynamics, which does not arise herein. However, being associated with the entropy s_2 , δQ represents an energy that may be transferred between different systems, and it will be shown herein to exhibit properties very similar to those of the usual heat. One should note that the confining potential makes the system anisotropic, and simulations were performed for $L_x \gg 2x_e$, where x_e corresponds to the cutoff in the probability distribution of Eq. (2.10). This indicates that in this stationary state vortices are present for $|x| < x_e$, and, consequently, no vortices are found close to the borders of the box in the x direction (see, e.g., Fig. 1 of Ref. [22], where a typical stationary state for a simulation in a box of $L_x = 280\lambda$ is presented). Hence, in order to conform with the simulations, the natural way to bring two such systems in contact is by joining together their sides L_x . Although no difficulties should appear by considering a contact between two systems with $L_{x,1} \neq L_{x,2}$ ($L_{x,1}, L_{x,2} \gg 2x_e$), herein we restrict ourselves to $L_{x,1} = L_{x,2} = L_x$, as shown in Fig. 1.

A. Thermal contact

The physical situation that we will analyze corresponds to two similar physical systems separated by a movable, rigid, and impermeable wall, as represented by the dashed line in Fig. 1. In order to simplify our analysis, we consider two systems characterized by the same values of f_0 and λ , so

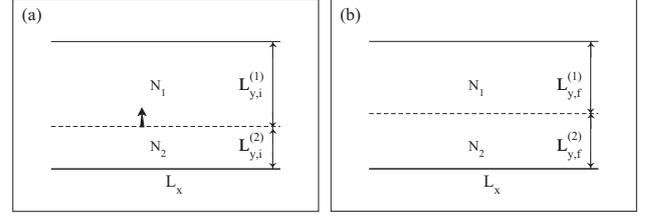


FIG. 1. A thermal contact may be achieved by considering a movable, rigid, and impermeable wall along the x direction (dashed line) separating two systems 1 and 2, characterized by N_1 and N_2 vortices, respectively. In case (a) a typical situation is represented where the net force exerted by the vortices of system 2 on the wall is greater than the one of the vortices in system 1, so the wall is pushed upwards. A mechanical equilibrium is reached in (b), where the forces from both sides cancel. The transformation is given by a change in the lengths in the y direction, from an initial state defined by lengths $(L_{y,i}^{(1)}, L_{y,i}^{(2)})$, to a final state defined by lengths $(L_{y,f}^{(1)}, L_{y,f}^{(2)})$.

the difference in their respective values of θ are due only to distinct values of n [cf. Eq. (2.4)]. Such a wall must feel the presence of magnetic vortices from both sides, and it may be constructed, e.g., by a double magnetic film. Since the forces among vortices are repulsive, the wall should be such as to experience a net repulsive force, being pushed from both sides. In Fig. 1(a) we represent a situation where the force due to system 2 is stronger than the one exerted by system 1, so the wall moves upwards, leading to a mechanical equilibrium shown in Fig. 1(b). The physical transformation consists in a change in the lengths in the y direction from a state with initial lengths $(L_{y,i}^{(1)}, L_{y,i}^{(2)})$ to a state with final lengths $(L_{y,f}^{(1)}, L_{y,f}^{(2)})$, which in Fig. 1 corresponds to

$$L_{y,f}^{(1)} = L_{y,i}^{(1)} - \delta L_y; \quad L_{y,f}^{(2)} = L_{y,i}^{(2)} + \delta L_y \quad (\delta L_y > 0), \quad (3.1)$$

in such a way to conserve their sum,

$$L_{y,f}^{(1)} + L_{y,f}^{(2)} = L_{y,i}^{(1)} + L_{y,i}^{(2)}. \quad (3.2)$$

In Fig. 1(a) we have considered a situation where $\theta_i^{(1)} < \theta_i^{(2)}$, which according to Eq. (2.4) corresponds to

$$\frac{N_1}{L_{y,i}^{(1)}} < \frac{N_2}{L_{y,i}^{(2)}}. \quad (3.3)$$

Although the precise form of the repulsive force exerted by a single vortex on the wall may vary according to the type of wall considered, we herein postulate that the intensity of the net force produced by a system of N vortices in a region of size L_y should depend, apart from characteristic constants, on the ratio N/L_y . This dependence is understood since the total force increases with the number of vortices, whereas the contribution from a single vortex decreases with its distance from the wall, leading to a decrease of the total force with L_y . Therefore, we assume that the intensity of the total force exerted by a system of vortices on the wall to be proportional to $\tilde{F}(N/L_y)$, where $\tilde{F}(x)$ represents a monotonically increasing function of x . In this way, the physical transformation occurs in such a way that the system with a higher density of vortices pushes the wall, leading to the equilibrium shown in Fig. 1(b),

characterized by

$$\frac{N_1}{L_{y,f}^{(1)}} = \frac{N_2}{L_{y,f}^{(2)}}, \quad (3.4)$$

which corresponds precisely to $\theta_f^{(1)} = \theta_f^{(2)} = \theta_f$.

Since s_2 is a state variable, one may use Eq. (2.14) to calculate the entropy changes for both systems 1 and 2 (to be referred herein as $\delta s_2^{(1)}$ and $\delta s_2^{(2)}$, respectively),

$$\frac{\delta s_2^{(1)}}{k} = \frac{1}{k} [s_{2,f}^{(1)} - s_{2,i}^{(1)}] = \frac{3^{2/3}}{5} \left[\left(\frac{\alpha \lambda^2}{k \theta_i^{(1)}} \right)^{1/3} - \left(\frac{\alpha \lambda^2}{k \theta_f} \right)^{1/3} \right], \quad (3.5)$$

$$\frac{\delta s_2^{(2)}}{k} = \frac{1}{k} [s_{2,f}^{(2)} - s_{2,i}^{(2)}] = \frac{3^{2/3}}{5} \left[\left(\frac{\alpha \lambda^2}{k \theta_i^{(2)}} \right)^{1/3} - \left(\frac{\alpha \lambda^2}{k \theta_f} \right)^{1/3} \right], \quad (3.6)$$

showing that $\delta s_2^{(1)} > \delta s_2^{(2)}$, for $\theta_i^{(1)} < \theta_i^{(2)}$ [or, equivalently, $\delta s_2^{(1)} < \delta s_2^{(2)}$, for $\theta_i^{(1)} > \theta_i^{(2)}$], similarly to what happens in standard thermodynamics, where the system at lower temperature absorbs heat (i.e., increases its entropy) from the one at higher temperature. Moreover, one sees that the entropy variations in Eqs. (3.5) and (3.6) present opposite signs only if the final temperature θ_f lies between the two initial temperatures, e.g., $\delta s_2^{(1)} > 0$ and $\delta s_2^{(2)} < 0$, if $\theta_i^{(1)} < \theta_f < \theta_i^{(2)}$.

In fact, a relation involving these temperatures may be found directly from Eq. (3.2). For that, one divides this equation by N_1 and multiplies the initial and final lengths of system 2 by unit [i.e., by (N_2/N_2)],

$$\frac{L_{y,f}^{(1)}}{N_1} + \frac{N_2}{N_1} \frac{L_{y,f}^{(2)}}{N_2} = \frac{L_{y,i}^{(1)}}{N_1} + \frac{N_2}{N_1} \frac{L_{y,i}^{(2)}}{N_2}. \quad (3.7)$$

Now one multiplies the equation above by the constant factor $k/(\pi f_0 \lambda^2)$ in order to use Eq. (2.4), so

$$\frac{1}{\theta_f} \left(1 + \frac{N_2}{N_1} \right) = \frac{1}{\theta_i^{(1)}} + \frac{N_2}{N_1} \frac{1}{\theta_i^{(2)}}, \quad (3.8)$$

from which one gets

$$\frac{1}{\theta_f} = \frac{N_1}{N_1 + N_2} \frac{1}{\theta_i^{(1)}} + \frac{N_2}{N_1 + N_2} \frac{1}{\theta_i^{(2)}}. \quad (3.9)$$

This equation may still be written in terms of the initial lengths of the two systems, $L_{y,i}^{(1)}$ and $L_{y,i}^{(2)}$. From Eq. (3.9) one has

$$\begin{aligned} \theta_f &= \frac{(N_1 + N_2) \theta_i^{(1)} \theta_i^{(2)}}{N_2 \theta_i^{(1)} + N_1 \theta_i^{(2)}} \\ &= \frac{[(N_1/L_{y,i}^{(1)}) + (N_2/L_{y,i}^{(1)})] \theta_i^{(1)} \theta_i^{(2)}}{(N_2/L_{y,i}^{(1)}) \theta_i^{(1)} + (N_1/L_{y,i}^{(1)}) \theta_i^{(2)}}. \end{aligned} \quad (3.10)$$

Now, multiplying the N_2 terms by unit [i.e., by $(L_{y,i}^{(2)}/L_{y,i}^{(2)})$], as well as both numerator and denominator by the constant factor $(\pi f_0 \lambda^2)/k$, one obtains

$$\theta_f = \frac{L_{y,i}^{(1)}}{L_{y,i}^{(1)} + L_{y,i}^{(2)}} \theta_i^{(1)} + \frac{L_{y,i}^{(2)}}{L_{y,i}^{(1)} + L_{y,i}^{(2)}} \theta_i^{(2)}. \quad (3.11)$$

One should call the attention to Eqs. (3.9) and (3.11), where the final equilibrium temperature is related to the two initial temperatures. In the first case one finds a relation between the temperature inverses, where each contribution in the right-hand side appears multiplied by the respective fraction of the total number of vortices. In the second case one has a linear relation between these temperatures, where each contribution in the right-hand side appears multiplied by its respective fraction of the total length occupied. It is important to note the similarity of the results above [particularly that of Eq. (3.11)] with the one corresponding to the exchange of heat between two substances A and B in standard thermodynamics; considering two substances with thermal capacities C_A and C_B , and at initial temperatures T_A and T_B , respectively, the final equilibrium temperature T_f attained after their thermal contact is given by [1]

$$T_f = \frac{C_A}{C_A + C_B} T_A + \frac{C_B}{C_A + C_B} T_B. \quad (3.12)$$

Both Eqs. (3.9) and (3.11) guarantee the most desirable result that the final equilibrium temperature θ_f must lie inside the interval defined by the two initial temperatures $\theta_i^{(1)}$ and $\theta_i^{(2)}$.

B. Thermal reservoir

Let us now address the situation of a contact in which system 1 presents a number of vortices much larger than the one of system 2, i.e., $N_1 \gg N_2$. This condition characterizes system 1 as a heat reservoir, a concept introduced qualitatively in Ref. [26]. In this case, considering $\theta_i^{(1)} < \theta_i^{(2)}$ in Eq. (3.9), one gets

$$\theta_f \cong \left(1 + \frac{N_2}{N_1} \right) \theta_i^{(1)}, \quad (3.13)$$

showing that the final equilibrium temperature should be slightly higher than the initial temperature of the reservoir, as expected. Below, we will explore this case further, with particular interest in the initial and final positions of the wall, defined by the transformation shown in Fig. 1. Hence, one has two possibilities satisfying Eq. (3.2), namely

$$L_{y,f}^{(1)} = L_{y,i}^{(1)} \mp \delta L_y; \quad L_{y,f}^{(2)} = L_{y,i}^{(2)} \pm \delta L_y \quad (\delta L_y > 0), \quad (3.14)$$

where the upper (lower) signs apply for a temperature decrease (increase) in system 2.

Dividing Eqs. (3.14) by $L_{y,f}^{(2)}$ and $L_{y,f}^{(1)}$, respectively, one obtains

$$\frac{L_{y,f}^{(1)}}{L_{y,f}^{(2)}} = \frac{L_{y,i}^{(1)}}{L_{y,i}^{(2)}} \mp \frac{\delta L_y}{L_{y,f}^{(2)}}, \quad (3.15)$$

$$\frac{L_{y,f}^{(2)}}{L_{y,f}^{(1)}} = \frac{L_{y,i}^{(2)}}{L_{y,i}^{(1)}} \pm \frac{\delta L_y}{L_{y,f}^{(1)}}. \quad (3.16)$$

Now we apply the condition that system 1 represents a thermal reservoir, so that Eq. (3.4) yields $(L_{y,f}^{(2)}/L_{y,f}^{(1)}) \ll 1$. Using this condition in Eq. (3.15), one concludes that both sides of this equation should be much larger than unit; the

same condition in Eq. (3.16) gives that both sides of this later equation are positive and much smaller than unit. Below we discuss the possible consequences.

(a) Upper signs: system 2 increases the length $L_{y,2}$, lowering its temperature.

Analyzing Eqs. (3.15) and (3.16) one concludes that

$$L_{y,i}^{(1)} \gg \delta L_y; \quad L_{y,f}^{(2)}, L_{y,i}^{(2)}, \delta L_y \ll L_{y,f}^{(1)}; \quad L_{y,i}^{(2)} \ll L_{y,i}^{(1)}, \quad (3.17)$$

where the last inequality was obtained from Eq. (3.2). This corresponds to a situation where the displacement of the wall is very small when compared with both lengths $L_{y,i}^{(1)}$ and $L_{y,f}^{(1)}$, leading to a negligible change in the temperature of system 1.

(b) Lower signs: system 2 decreases its length $L_{y,2}$, increasing its temperature.

From Eq. (3.16) one obtains that $L_{y,i}^{(2)} \gtrsim \delta L_y$, leading to two possibilities. The first one consists in $L_{y,i}^{(1)}, L_{y,f}^{(1)} \gg \delta L_y$, resulting in a situation similar to the one described by Eq. (3.17), in agreement with the concept of a heat reservoir for system 1. This second possibility corresponds to a situation where the effective temperature of the reservoir changes and so it will not be addressed herein.

C. Zeroth law

Let us consider two systems (1 and 2) in contact, as shown in Fig. 1, for which their equilibrium is characterized by

$$\frac{N_1}{L_{y,1}} = \frac{N_2}{L_{y,2}}. \quad (3.18)$$

Now, by putting a third system in contact and in equilibrium with system 1, one has

$$\frac{N_1}{L_{y,1}} = \frac{N_3}{L_{y,3}}. \quad (3.19)$$

Hence, the right-hand sides of Eqs. (3.18) and (3.19) are equal to one another, so systems 2 and 3 are also in equilibrium, i.e., $(N_2/L_{y,2}) = (N_3/L_{y,3})$.

This result characterizes 1 as a test system, i.e., a thermometer: All systems in equilibrium with 1 are in equilibrium with each other. These systems have a property in common, which is their effective temperature θ , defined in Eq. (2.4). Therefore, one can formulate the zeroth principle for the present system:

Two systems of superconducting vortices for which thermal effects are negligible in comparison with those associated with their effective temperatures are said to be in thermal equilibrium if, being in contact, no heat flows in either way. The zeroth principle can be enunciated by stating that two systems in thermal equilibrium with a third one are in thermal equilibrium with each other. This corresponds to the transitivity of their effective temperatures.

IV. TRANSFORMATIONS AND CYCLES

In this section we will analyze the possible physical transformations that follow from Eq. (2.17), and, considering these transformations, we will construct two cycles, namely those equivalents to the Carnot refrigerator and Otto cycle. Although the Carnot cycle has already been studied in

Ref. [26], herein we will show that considering this cycle in its reverse way, a similar to the Carnot refrigerator follows, with the same form for its coefficient of performance. However, the efficiency obtained for the Otto cycle presents a different (i.e., generalized) form, showing the the Carnot cycle and refrigerator are special machines within the context of a consistent thermodynamical framework.

A. Physical transformations

From Eq. (2.17) one can define four possible physical transformations, namely isothermic ($\theta = \text{constant}$), adiabatic ($s_2 = \text{constant}$), and two additional ones, to be called herein iso- α and iso- σ . However, from Eqs. (2.14), (2.15), and (2.19), one notices that an adiabatic process corresponds to one of the conditions,

$$\frac{\alpha}{\theta} = \text{const}; \quad \sigma = \frac{u}{\alpha} = \text{const}, \quad (4.1)$$

or properly defined combinations of them. Therefore, in the present system, the adiabatic and iso- σ transformations correspond precisely to the same process. Lets us then consider the three possible transformations associated with Eq. (2.17), occurring from an initial state i to a final state f .

1. Adiabatic (or iso- σ) transformation

The total work done *on* the system in an adiabatic transformation from an initial state characterized by (θ_i, α_i) , to a final state with (θ_f, α_f) , is given by

$$u_f - u_i = W = \int_{\alpha_i}^{\alpha_f} \sigma d\alpha = \sigma(\alpha_f - \alpha_i), \quad (4.2)$$

where

$$\sigma = \frac{3^{2/3}}{10} \lambda^2 \left(\frac{k\theta_i}{\alpha_i \lambda^2} \right)^{2/3} = \frac{3^{2/3}}{10} \lambda^2 \left(\frac{k\theta_f}{\alpha_f \lambda^2} \right)^{2/3}. \quad (4.3)$$

Hence, in an adiabatic process one gets positive (negative) work for $\alpha_f > \alpha_i$ ($\alpha_f < \alpha_i$).

2. Isothermal transformation

For an isothermal process at a temperature θ , one has

$$\begin{aligned} Q &= \int_{s_{2,i}}^{s_{2,f}} \theta ds_2 = \theta(s_{2,f} - s_{2,i}) \\ &= \frac{3^{2/3}}{5} (k\theta)^{2/3} [(\alpha_i \lambda^2)^{1/3} - (\alpha_f \lambda^2)^{1/3}], \end{aligned} \quad (4.4)$$

$$\begin{aligned} W &= \int_{\alpha_i}^{\alpha_f} \sigma d\alpha = \frac{3^{2/3}}{10} \lambda^2 \left(\frac{k\theta}{\lambda^2} \right)^{2/3} \int_{\alpha_i}^{\alpha_f} \alpha^{-2/3} d\alpha \\ &= \frac{3^{5/3}}{10} (k\theta)^{2/3} [(\alpha_f \lambda^2)^{1/3} - (\alpha_i \lambda^2)^{1/3}], \end{aligned} \quad (4.5)$$

$$u_f - u_i = Q + W = \frac{3^{2/3}}{10} (k\theta)^{2/3} [(\alpha_f \lambda^2)^{1/3} - (\alpha_i \lambda^2)^{1/3}]. \quad (4.6)$$

Then, for the isothermal process one has positive (negative) work and variation of internal energy, whereas the system releases (absorbs) heat, if $\alpha_f > \alpha_i$ ($\alpha_f < \alpha_i$).

3. Iso- α transformation

These transformations were considered in the previous section in the analysis of the thermal contact between two systems. They are characterized by $du = \delta Q = \theta ds_2$, where

$$u_f - u_i = \frac{3^{2/3}}{10} (\alpha \lambda^2)^{1/3} [(k\theta_f)^{2/3} - (k\theta_i)^{2/3}], \quad (4.7)$$

$$s_{2,f} - s_{2,i} = \frac{3^{2/3} k}{5} \left[\left(\frac{\alpha \lambda^2}{k\theta_i} \right)^{1/3} - \left(\frac{\alpha \lambda^2}{k\theta_f} \right)^{1/3} \right]. \quad (4.8)$$

Considering the above-defined transformations, one can construct cycles, and two of them will be discussed below, namely the Carnot refrigerator and the Otto cycle. In Ref. [26] a cycle analogous to a Carnot cycle was studied, and its efficiency was shown to follow the standard form, i.e., $\eta = 1 - (\theta_2/\theta_1)$, where θ_1 and θ_2 represent the effective temperatures associated with the isothermal transformations of the cycle, with $\theta_1 > \theta_2$. For completeness, herein we consider the reversed Carnot cycle, i.e., the Carnot refrigerator. In contrast to the Carnot cycle, which keeps the same form for its efficiency, we will show below that the efficiency of the Otto cycle changes its form, now being given by a nonlinear function of the associated temperatures.

B. Carnot refrigerator

Let us now define a system analogous to the reversed Carnot cycle by considering two isothermal and two adiabatic processes, intercalated, as illustrated in Fig. 2 in a plane σ/λ^2 (dimensionless) versus $\alpha\lambda^2$ (dimensions of energy). The main properties of this cycle are described below.

(i) An amount of heat Q_2 is absorbed in the isothermal process at the lower temperature θ_2 , whereas the system releases heat Q_1 in the isothermal process at the higher temperature θ_1 . For the complete cycle, the variation of internal energy is zero and so $Q_1 = W + Q_2$ (considering Q_1 , Q_2 , and W all positive). It is important to note our definition of the work contribution in Eq. (2.17) (associated with the plus sign).

(ii) In a plot σ versus α (or, equivalently, σ/λ^2 versus $\alpha\lambda^2$, as in Fig. 2), the work associated with a given process corresponds to the area below such transformation. As shown above, work is positive (negative) for transformations that increase (decrease) α . Therefore, the total work done on the system, calculated as $W = W_{ab} + W_{bc} + W_{cd} + W_{da}$, is given by the area enclosed in the cycle of Fig. 2.

(iii) Manipulating Eqs. (2.19) and (4.4), one obtains the well-known result relating the two isothermal processes,

$$\frac{Q_1}{Q_2} = \frac{\theta_1}{\theta_2}. \quad (4.9)$$

(iv) Defining the coefficient of performance of the refrigerator in the standard way [1,2], one gets

$$K = \frac{Q_2}{W} = \frac{Q_2}{Q_1 - Q_2} = \frac{\theta_2}{\theta_1 - \theta_2}. \quad (4.10)$$

Noting that $K = Q_2/(eQ_1)$, where e represents the efficiency of the Carnot engine operating in the reverse way with respect to the one of Fig. 2, one recovers the celebrated efficiency of

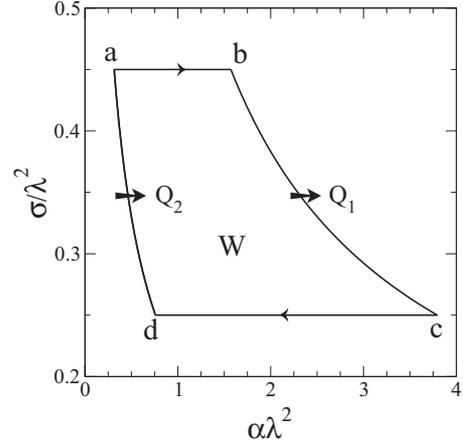


FIG. 2. The Carnot refrigerator for a system of interacting vortices under overdamped motion by neglecting thermal noise. The abscissa $\alpha\lambda^2$ presents dimensions of energy (e.g., Joules), whereas the ordinate σ/λ^2 is dimensionless. The transformations for σ constant are adiabatic, and herein they were chosen to occur for $(\sigma/\lambda^2) = 0.45$ ($a \rightarrow b$) and $(\sigma/\lambda^2) = 0.25$ ($c \rightarrow d$). The isothermal transformations are characterized by $\sigma \sim \alpha^{-2/3}$ [cf. Eq. (2.19)] and they occur for $k\theta_1 = 5$ (units of energy) in $b \rightarrow c$, and $k\theta_2 = 1$ (units of energy) in $d \rightarrow a$, i.e., $\theta_1 > \theta_2$. Operating as a refrigerator, an amount of heat Q_2 is absorbed at θ_2 and Q_1 is released at θ_1 ; as shown in text, the area inside the cycle represents the total work W done on the refrigerator with $Q_1 = W + Q_2$ (conventionalizing all these three quantities as positive). The cycle above holds for any system of units, e.g., by considering all quantities with dimensions of energy in Joules.

the Carnot cycle,

$$e = \frac{W}{Q_1} = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{\theta_2}{\theta_1}. \quad (4.11)$$

These results, together with those of Ref. [26], show the appropriateness of the effective-temperature definition proposed in Ref. [25], as well as of the fundamental relation of Eq. (2.17).

C. Otto cycle

Let us now analyze a cycle analogous to the Otto cycle which consists of two adiabatic and two isovolumetric transformations, intercalated, representing a rough approximation of a gasoline engine [2,36]. In the present case, the isovolumetric transformations are replaced by the above-defined iso- α processes. The main properties of this cycle are described below.

(i) In the plane σ/λ^2 versus $\alpha\lambda^2$ this cycle is represented by a rectangle, as shown in Fig. 3. The effective temperature θ changes along all four transformations, presenting the values θ_a , θ_b , θ_c , and θ_d , at the vertices of the rectangle.

(ii) An amount of heat Q_1 is absorbed in the iso- α transformation $a \rightarrow b$, whereas the system releases heat Q_2 in the iso- α transformation $c \rightarrow d$. The total work done on the system is given by the area enclosed in the cycle of Fig. 3, being negative, as expected from Eq. (2.17). If one defines $\mathcal{W} = -W$ as the work done by the system, the variation of internal energy is zero for the complete cycle, and one has

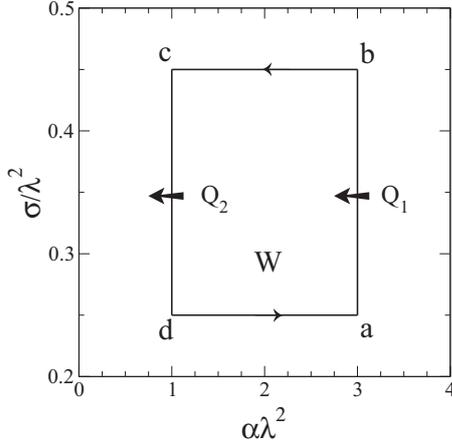


FIG. 3. The Otto cycle for a system of interacting vortices under overdamped motion by neglecting thermal noise. The abscissa $\alpha\lambda^2$ presents dimensions of energy (e.g., Joules), whereas the ordinate σ/λ^2 is dimensionless. The transformations for σ constant are adiabatic, and herein they were chosen to occur for $(\sigma/\lambda^2) = 0.45$ ($b \rightarrow c$) and $(\sigma/\lambda^2) = 0.25$ ($d \rightarrow a$). The iso- α transformations were chosen to occur for $\alpha\lambda^2 = 3.0$ (units of energy) ($a \rightarrow b$) and $\alpha\lambda^2 = 1.0$ (units of energy) ($c \rightarrow d$), respectively. An amount of heat Q_1 is absorbed at transformation $a \rightarrow b$, whereas Q_2 is released at $c \rightarrow d$. The area inside the cycle represents the total work W done on the system, which is negative, as expected from Eq. (2.17). The abscissa $\alpha\lambda^2$ presents dimensions of energy, whereas the ordinate σ/λ^2 is dimensionless; the cycle above holds for any system of units, e.g., one may consider all quantities with dimensions of energy in Joules.

$Q_1 = W + Q_2$ (conventionalizing all these three quantities as positive).

(iii) Considering the equation of state [Eq. (2.19)] and the property for the adiabatic transformation, $(\alpha/\theta) = \text{const}$, one concludes that $\theta_b > \theta_a > \theta_d$ and $\theta_b > \theta_c > \theta_d$, so θ_b and θ_d represent the highest and lowest temperatures of the cycle, respectively. An inequality involving θ_c and θ_a depends on the particular choices of coordinates for the vertices of the rectangle; the case shown in Fig. 3 corresponds to $\theta_c \approx 0.80\theta_a$.

(iv) Using Eq. (4.7) one can calculate Q_1 and Q_2 ,

$$Q_1 = \frac{3^{2/3}}{10} (\alpha_a \lambda^2)^{1/3} [(k\theta_b)^{2/3} - (k\theta_a)^{2/3}], \quad (4.12)$$

$$Q_2 = \frac{3^{2/3}}{10} (\alpha_c \lambda^2)^{1/3} [(k\theta_c)^{2/3} - (k\theta_d)^{2/3}], \quad (4.13)$$

which are both positive quantities, with $Q_1 > Q_2$ by definition. In the adiabatic transformation $b \rightarrow c$ one has that $(\alpha_c/\alpha_b) = (\theta_c/\theta_b)$, and using that $\alpha_b = \alpha_a$, one obtains the efficiency of the cycle,

$$\eta = 1 - \frac{Q_2}{Q_1} = 1 - \frac{\theta_d}{\theta_a} \frac{(\theta_c/\theta_d)^{2/3} - 1}{(\theta_b/\theta_a)^{2/3} - 1}. \quad (4.14)$$

It is important to mention that for the standard Otto cycle with an ideal gas as the working fluid, the efficiency η differs from the expression of Eq. (4.14) through a replacement of the exponents $2/3$ by unit [36].

Consequently, we conclude that for a system of superconducting vortices the Carnot cycle is also very special, being characterized by an efficiency that has precisely the same form as the one found in standard thermodynamics. The present results, together with those of Refs. [25,26], show that the definition of effective temperature, as well as of the fundamental relation in Eq. (2.17), are very appropriate within the present context. Since the Carnot cycle represents a fundamental system in standard thermodynamics, from which the absolute scale of temperature may be constructed, these results suggest that a complete thermodynamical formalism should also hold herein. In what follows, we will explore Eq. (2.17) by performing Legendre transformations in order to define thermodynamic potentials, deriving further important relations from them.

V. POTENTIALS

A. Internal energy

From Eq. (2.17) one has that the internal energy depends on the pair of independent variables (s_2, α) , i.e., $u = u(s_2, \alpha)$. Manipulating Eqs. (2.13) and (2.14) [or, equivalently, inverting Eq. (2.15)] one gets

$$u(s_2, \alpha) = \frac{9}{250} \frac{\alpha\lambda^2}{(1 - s_2/k)^2}. \quad (5.1)$$

From the equation above one obtains, respectively, the equivalent to Eq. (2.16), as well as the equation of state in Eq. (2.19),

$$\left(\frac{\partial u}{\partial s_2} \right)_\alpha = \theta; \quad \left(\frac{\partial u}{\partial \alpha} \right)_{s_2} = \sigma. \quad (5.2)$$

Now, since $u(s_2, \alpha)$ is a state function, its second derivatives must be independent of the order of differentiation, leading to the following Maxwell relation:

$$\frac{\partial^2 u}{\partial \alpha \partial s_2} = \frac{\partial^2 u}{\partial s_2 \partial \alpha} \Rightarrow \left(\frac{\partial \sigma}{\partial s_2} \right)_\alpha = \left(\frac{\partial \theta}{\partial \alpha} \right)_{s_2}. \quad (5.3)$$

B. Helmholtz free energy

Let us now introduce the Helmholtz free energy $f(\theta, \alpha)$ by means of the Legendre transformation,

$$f(\theta, \alpha) = u - \theta s_2; \quad \Rightarrow df = -s_2 d\theta + \sigma d\alpha. \quad (5.4)$$

One obtains the following free energy:

$$f(\theta, \alpha) = \frac{3^{5/3}}{10} (\alpha\lambda^2)^{1/3} (k\theta)^{2/3} - k\theta, \quad (5.5)$$

which satisfies the following relations:

$$\left(\frac{\partial f}{\partial \theta} \right)_\alpha = -s_2; \quad \left(\frac{\partial f}{\partial \alpha} \right)_\theta = \sigma. \quad (5.6)$$

From its second derivatives, the corresponding Maxwell relation appears,

$$\frac{\partial^2 f}{\partial \alpha \partial \theta} = \frac{\partial^2 f}{\partial \theta \partial \alpha} \Rightarrow \left(\frac{\partial s_2}{\partial \alpha} \right)_\theta = - \left(\frac{\partial \sigma}{\partial \theta} \right)_\alpha. \quad (5.7)$$

C. Gibbs free energy

Herein we define the Gibbs potential $g(\theta, \sigma)$ through

$$\begin{aligned} g(\theta, \sigma) &= f - \sigma\alpha = u - \theta s_2 - \sigma\alpha; \\ \Rightarrow dg &= -s_2 d\theta - \alpha d\sigma. \end{aligned} \quad (5.8)$$

From the potentials above one gets,

$$g(\theta, \sigma) = k\theta \left[\frac{6}{10^{3/2}} \frac{\lambda}{\sigma^{1/2}} - 1 \right], \quad (5.9)$$

which satisfies

$$\left(\frac{\partial g}{\partial \theta} \right)_\sigma = -s_2; \quad \left(\frac{\partial g}{\partial \sigma} \right)_\theta = -\alpha. \quad (5.10)$$

The corresponding Maxwell relation is

$$\left(\frac{\partial s_2}{\partial \sigma} \right)_\theta = \left(\frac{\partial \alpha}{\partial \theta} \right)_\sigma. \quad (5.11)$$

D. Enthalpy

As mentioned before, comparing Eqs. (2.13) and (2.19) one notices that $u = \sigma\alpha$, representing a peculiarity of the present system. In order to construct a potential of the type $h = h(s_2, \sigma)$ one should define

$$\begin{aligned} h(s_2, \sigma) &= u - \sigma\alpha = f + \theta s_2 - \sigma\alpha = g + \theta s_2; \\ \Rightarrow dh &= \theta ds_2 - \alpha d\sigma. \end{aligned} \quad (5.12)$$

Hence, this particular potential presents an unusual behavior, given by a trivial enthalpy, $h = 0$. A consequence of this result has already appeared in the previous section, where we have shown that an adiabatic transformation is given by $ds_2 = 0$, or, equivalently, by $d\sigma = 0$, implying that the present system cannot exchange ‘‘heat’’ (i.e., it cannot vary its entropy) for σ fixed. Therefore, a complete description is given in terms of the three previously defined potentials, namely internal energy, $u(s_2, \alpha)$, and free energies, $f(\theta, \alpha)$ and $g(\theta, \sigma)$; the enthalpy $h(s_2, \sigma)$ does not contain any new information. Similar anomalies may be found also on other physical systems, within BG statistical mechanics, as described below.

(a) In the three-dimensional ideal gas, for which $pv = 2u/3$ (valid for the classic case, as well as in both quantum statistics [3]), leading to an enthalpy $h = 5u/3$, showing that in this case, the enthalpy does not represent an independent thermodynamic potential.

(b) In the ideal paramagnet, where the internal energy is given in terms of the magnetization M and external magnetic field H as $u = -MH$ [1,2]. Such a relation also leads to a simple enthalpy, which does not carry any new information about the system. Note that, for magnetic systems, the first law may be written in two distinct ways (depending on the definition of work, $\delta W = -MdH$ or $\delta W = HdM$) and one has the following possibilities for the enthalpy of the ideal paramagnet: (i) For the first law written as $du = Tds - MdH$, the enthalpy is defined as $h = u + MH = 0$, leading to a situation analogous to the one found herein, and (ii) in the case $du = Tds + HdM$, one considers the Legendre transformation $h = u - MH = -2u$, and so the enthalpy is essentially given in terms of the internal energy, and it does not

represent an independent thermodynamic potential (similarly to what happens for the three-dimensional ideal gas).

VI. RESPONSE FUNCTIONS

In this section we define quantities analogous to the response functions of standard thermodynamics. In Ref. [25] we have already introduced the specific heat for a fixed α , calculated in three different ways, namely, from the internal energy, or the entropy, or the free energy,

$$c_\alpha = \left(\frac{\partial u}{\partial \theta} \right)_\alpha = \theta \left(\frac{\partial s_2}{\partial \theta} \right)_\alpha = -\theta \left(\frac{\partial^2 f}{\partial \theta^2} \right)_\alpha = \frac{3^{2/3}}{15} k \left(\frac{\alpha \lambda^2}{k\theta} \right)^{1/3}, \quad (6.1)$$

leading to $c_\alpha \geq 0$. In a similar way, one can define c_σ ,

$$c_\sigma = \theta \left(\frac{\partial s_2}{\partial \theta} \right)_\sigma = \left(\frac{\partial h}{\partial \theta} \right)_\sigma = 0, \quad (6.2)$$

where we have used Eq. (5.12), in addition to the fact that $h = 0$. This is an expected result since we have seen in the previous sections that this system cannot exchange heat for σ fixed. One should notice the contrast with general substances in thermodynamics for which $c_p \geq c_v$.

Now we consider standard calculations, in order to define other response functions for this system [1,2]. In the equation of state [Eq. (2.19)] one has $\sigma = \sigma(\theta, \alpha)$ so

$$d\sigma = \left(\frac{\partial \sigma}{\partial \theta} \right)_\alpha d\theta + \left(\frac{\partial \sigma}{\partial \alpha} \right)_\theta d\alpha, \quad (6.3)$$

or, equivalently, rewriting the equation of state as $\alpha = \alpha(\theta, \sigma)$, one gets

$$d\alpha = \left(\frac{\partial \alpha}{\partial \theta} \right)_\sigma d\theta + \left(\frac{\partial \alpha}{\partial \sigma} \right)_\theta d\sigma. \quad (6.4)$$

Let us then define the following quantities, corresponding, respectively, to the coefficient of expansion and isothermal compressibility of standard thermodynamics:

$$\gamma = \frac{1}{\alpha} \left(\frac{\partial \alpha}{\partial \theta} \right)_\sigma; \quad \kappa = -\frac{1}{\alpha} \left(\frac{\partial \alpha}{\partial \sigma} \right)_\theta, \quad (6.5)$$

which measure changes in α with respect to variations in θ (σ fixed) and σ (θ fixed). By analyzing the equation of state, one sees that these quantities are both positive; they will be calculated later. Considering the pair of independent variables (θ, σ) , one has

$$\theta ds_2 = \theta \left[\left(\frac{\partial s_2}{\partial \theta} \right)_\sigma d\theta + \left(\frac{\partial s_2}{\partial \sigma} \right)_\theta d\sigma \right] = \theta \left(\frac{\partial s_2}{\partial \sigma} \right)_\theta d\sigma, \quad (6.6)$$

where we have used Eq. (6.2) in the last equality. Substituting Eq. (6.3) for α constant in the equation above, the quantity c_α in Eq. (6.1) may be written as

$$c_\alpha = \theta \left(\frac{\partial s_2}{\partial \sigma} \right)_\theta \left(\frac{\partial \sigma}{\partial \theta} \right)_\alpha. \quad (6.7)$$

Now, using Eq. (6.4) for α constant and the definitions of Eq. (6.5), one obtains

$$\left(\frac{\partial \sigma}{\partial \theta} \right)_\alpha = \frac{\gamma}{\kappa}. \quad (6.8)$$

The equation above, together with the Maxwell relation of Eq. (5.11), allows us to write Eq. (6.7) as

$$c_\alpha = \alpha\theta \frac{\gamma^2}{\kappa}, \quad (6.9)$$

which may be compared with $c_p - c_v = vT\gamma^2/\kappa$ of standard thermodynamics [1,2]. Since the variables θ and α are positive, c_α and κ should present the same sign; indeed, considering the equation of state, one may calculate the quantities of Eq. (6.5),

$$\gamma = \frac{1}{\theta}; \quad \kappa = \frac{3}{2\sigma}, \quad (6.10)$$

showing that γ and κ are both positive quantities for the present system, as expected. Herein, one should note the similar quantities for the ideal gas, $\gamma = 1/T$ and $\kappa = 1/p$ [1]. Substituting the results above in Eq. (6.9), one recovers the expression for c_α shown in Eq. (6.1).

Therefore, the quantities introduced above, c_α , γ , and κ , behave very similarly to the corresponding ones of standard thermodynamics, including the relation between them, Eq. (6.9). The contrast is given by a zero c_σ , which is a direct consequence of the fact that transformations with $\sigma = \text{const}$ and adiabatics are precisely the same or, equivalently, of the unusual enthalpy, $h = 0$.

VII. CONCLUSIONS

We have introduced a framework—analogs to standard thermodynamics—for a system of interacting vortices under overdamped motion, which is currently used as a relevant model for type II superconductors. The concepts explored are based on an effective temperature θ defined previously, which was recently shown to be experimentally controllable and to present typical values in type II superconductors that are much larger than typical temperatures of superconducting phases, i.e., $T/\theta \simeq 0$ [25]. Hence, thermal effects were neglected, and only those related to θ were investigated.

The temperature θ is conjugated to Tsallis generalized entropy s_q , with $q = 2$, leading to a heat contribution, $\delta Q = \theta ds_2$. We have explored this heat contribution by proposing an specific form of thermal contact between two such systems, and defining the corresponding equilibrium condition as a physical situation where no heat exchange takes place. We have verified that the exchange of heat occurs similarly to the one of standard thermodynamics, being characterized by: (i) The system at lower temperature absorbs heat from the one at higher temperature; (ii) The final equilibrium temperature θ_f lies between the two initial temperatures. Based on these results we have enunciated the zeroth principle for these systems.

The consistency of a form analogous to the first-law of thermodynamics, $du = \theta ds_2 + \delta W$, was verified lately by means of a Carnot cycle, whose efficiency was shown to be $\eta = 1 - (\theta_2/\theta_1)$, where θ_1 and θ_2 represent the effective temperatures associated with the isothermal transformations of the cycle, with $\theta_1 > \theta_2$ [26]. Herein, we have illustrated further this first-law proposal by analyzing the Carnot refrigerator, as well as the Otto cycle. We have shown that the latter cycle presents an expression for its efficiency which differs from that of usual thermodynamics. Consequently, the Carnot cycle

(as well as the Carnot refrigerator) appears to be universal, in the sense that its efficiency keeps the same form as the one of standard thermodynamics, thus allowing the possibility for constructing a temperature scale associated with θ .

We have consolidated this first-law proposal by following the usual procedure for obtaining different potentials, i.e., applying Legendre transformations for distinct pairs of independent variables. From these potentials we have derived the equation of state, Maxwell relations, and defined response functions. All results presented are shown to be consistent with those of standard thermodynamics for $T > 0$. As a curious aspect, we have found the associated enthalpy as identically zero; this is a peculiarity of the present system, attributed to its equation of state, analogously to what happens in the usual thermodynamical description of an ideal paramagnet.

One should call the attention to one aspect of the present approach, concerning the dependence on N (total number of particles) of the quantities analyzed herein. The present results follow from a nonlinear Fokker-Planck equation, and more particularly, from its stationary-state solution, $P_{\text{st}}(x)$. As usual in any treatment using a Fokker-Planck equation, the associated probability distribution refers to a single particle of the system, and particularly, the present nonlinear diffusion contribution represents the effect of $N - 1$ vortices on a tagged vortex. Based on this, all physical quantities derived from $P_{\text{st}}(x)$, like u and s_2 , correspond to one-vortex properties. Recent analyses (see, e.g., Ref. [37]) suggest that the total entropy of a given system should be considered on the same footing as the number of elements N and volume V , i.e., it should be an extensive quantity; based on this, the total entropy of the present system becomes $S_2 = Ns_2$. Consequently, other quantities appearing in the fundamental thermodynamic relations (e.g., the first law, $du = \theta ds_2 + \sigma d\alpha$ and Legendre transformations) should present scaling functions of N such as to preserve the extensivity of the total entropy S_2 . In order to validate such an important proposal in the present system, one should investigate the problem through other techniques, like numerical simulations.

To conclude, we have presented a consistent thermodynamic framework for a system where thermal effects may be neglected, and only those related to an effective temperature θ were investigated. Although the Boltzmann-Gibbs entropy becomes zero in this limit, the entropy s_2 of nonextensive statistical mechanics appears as the appropriate conjugated variable to θ . The present results give further support for the system considered herein as an important physical application for nonextensive statistical mechanics and certainly, measurements in type II superconductors turn out to be highly desirable in order to confirm the validity of these theoretical predictions. Although we have focused our analysis to interacting type II superconducting vortices, the present results may be extended to other systems of interacting particles, with similar characteristics.

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