A note on the connection between the Tsallis' thermodynamics and cumulative prospect theory

B. A. Mello*
D. O. Cajueiro**

* Institute of Physics – University of Brasilia, 70910-900 DF Brazil.

** Department of Economics – University of Brasilia, 70910-900 DF Brazil and National Institute of Science and Technology for Complex Systems.

Abstract: This note presents explicitly a strong connection between the Tsallis thermodynamics and the so-called prospect theory introduced to take into account agent's decisions under risk. Moreover, using the cumulative prospect theory adapted here for a continuous framework, we show that if the Tsallis parameter q belongs to the interval [0,1], then the prospect theory requirement that extreme events are overvalued is satisfied.

Keywords: Prospect theory; Tsallis' Thermodynamics.

JEL Codes: C02.

1 Introduction

In this decade, one of the many attempts of statistical physics has been to apply well established techniques and concepts to cope with economic and financial phenomena. In spite of the wide spectrum of focus of the so-called econophysics, one of the well succeeded fields of investigation has been to seek for connections between the Tsallis thermodynamics introduced in Tsallis (1988) and the economic and econometric theory. One may find overviews of these attempts in Tsallis *et al*. (2003) and Tsallis (2009). In Anteneodo *et al.* (2002), the first connection between the Tsallis statistics and the Prospect Theory (PT) is presented. In Borland (2002), an extension of the celebrated Black-Scholes model (Black and Scholes 1973) is provided. In particular, the Black Scholes model is recovered when $q \to 1$. In Queirós (2004), the ARCH time series model is related to the non-extensive statistical mechanics. In Cajueiro (2005), the q-exponential function well known in the deformed algebra inspired in the Tsallis's nonextensive thermodynamics is used to model discount functions in intertemporal choices which present the phenomenon known as increasing patience. In this case, the Tsallis parameter q can be seen as a measure of consistency in intertemporal choices. In Takahashi *et al.*(2007), the model introduced in Cajueiro (2005) is tested empirically. Other connections between the Tsallis thermodynamics and decision theory may be found in Takahashi (2007a), Takahashi (2007b), Takahashi (2008), Takahashi *et al.* (2008), and Takahashi (2009). Finally, Martinez *et al*. (2008) consider the one-parameter generalizations of the logarithmic and exponential that arise in the context of non-extensive thermostatistics and show that these functions are suitable to describe and unify the great majority of continuous growth models.

In this note, we extend the Cumulative PT introduced in Tversky and Kahneman (1992) for a continuous framework to show mathematically a strong relationship between this theory and the Tsallis statistics.

This note proceeds as follows. In section 2, the Tsallis distribution is revised. Section 3 presents the main ideas of the PT. In section 4, our main results are presented. Finally, section 5 presents the conclusions of this work.

2 The Tsallis distribution

One of the most important contributions brought by non-extensive thermodynamics was the so-called non-extensive distributions[1]. The most common is the one that is found from the maximization of the non-extensive entropy (Tsallis, 1988)

$$S_q = k\left(\frac{1-\int p(x)^q\,dx}{q-1}\right) \quad (1)$$

subjected to the usual constraints

$$\int p(x)dx = 1 \quad (2)$$

$$< x-\bar{x} > \equiv \int (x-\bar{x})p(x)^q\,dx = 0 \quad (3)$$

$$< (x-\bar{x})^2 > \equiv \int (x-\bar{x})^2 p(x)^q\,dx = \sigma_q^2 \quad (4)$$

[1] See Borges (2004) for a comprehensive presentation of these distributions.

i.e.,

$$p(x) = \frac{1}{z_q}(1 + \beta_q (q-1)(x-\bar{x})^2)^{-\frac{1}{q-1}} \tag{5}$$

where

$$z_q = \left(\frac{\pi}{\beta_q (q-1)}\right)^{1/2} \frac{\Gamma((3-q)/2(q-1))}{\Gamma(1/(q-1))} \tag{6}$$

is the appropriate normalization factor.

If $q = 1$, equation (5) recovers the gaussian distribution.

3 The prospect theory

The PT was introduced in Kahneman and Tversky (1970) to circumvent the descriptive failures of expected utility theory of decision making, for instance: (1) The rational theory of choice assumes that equivalent formulations of a choice problem should give rise to the same preference order. However, there is much evidence contrary to this assumption. (2) According to the expected principle, the utility of a risky prospect is linear in outcome probabilities. Nonetheless, there are several examples of choice problems that do not support this assumption. (3) People's wish to bet on uncertain events depend not only on the degree of uncertainty but also on its source. (4) Risk aversion is the general assumption in economic and decision theory. However, this assumption is not consistent with people's preference for a small probability of winning a large prize over the expected value of that prospect and the choice between a sure loss and a substantial probability of a large loss.

In fact, PT differs from expected utility theory in at least two important points: (1) While the expected utility theory handles the probabilities attached to particular outcomes, the PT treats preferences as a function of "decision weight", which do not always correspond to probabilities, but a function of them. Specifically, PT postulates that decision weights tend to overweight small probabilities and underweight moderate and high probabilities. (2) PT also replaces the notion of "utility function" with "value". Whereas utility is usually defined only in terms of net wealth, value is defined in terms of gains and losses (deviations from a reference point). The value function has a different shape for gains and losses. For losses it is convex and relatively steep, for gains it is concave and not quite so steep.

4 Results

One remarkable aspect of the non-extensive statistics is that the weighting function used to evaluate the expected values is not the probability (Tsallis, 1988) but a power of it. In that theory, the expected value of a function *v(x)* is

$$E(v) = \int_{-\infty}^{\infty} v(x) p(x)^q \, dx, \tag{7}$$

with *p(x)* given by equation (5). Interpreting the meaning of the above expression is one of the main sources of disputes around the non-extensive statistics. As realized by Tsallis and others (Anteneodo et al., 2002) that property is shared by the PT (Kahneman and Tversky, 1970).

Following the *Cumulative Prospect Theory* (Tversky and Kahneman, 1992), we write down here[2] the equivalent continuous version of the PT where the investor's expectation can be approximated by

$$E(v)=\int_{-\infty}^{0} v(x)\pi^{-}[A^{-}(x)]\,dx+\int_{0}^{\infty} v(x)\pi^{+}[A^{+}(x)]\,dx. \tag{8}$$

In the above equation, A^{-} and A^{+} are, respectively, the cumulative probability function of negative and positive outcomes:

$$A^{-}(x)=\int_{-\infty}^{x} P(x')\,dx', \quad A^{+}(x)=\int_{x}^{\infty} P(x')\,dx'. \tag{9}$$

The decision weights $\pi^{\pm}(A^{\pm})$ are defined as the derivative of a capacity function

$$\pi^{-}[A^{-}(x)]=\frac{dw^{-}[A^{-}(x)]}{dx}=\left.\frac{dw^{-}[A^{-}]}{dA^{-}}\right|_{A^{-}(x)} P(x), \tag{10}$$

$$\pi^{+}[A^{+}(x)]=-\frac{dw^{+}[A^{+}(x)]}{dx}=\left.\frac{dw^{+}[A^{+}]}{dA^{+}}\right|_{A^{+}(x)} P(x). \tag{11}$$

Values of $dw^{\pm}/dA^{\pm}>1$ correspond to overestimated outcomes and vice versa.

The connection between the Tsallis statistics is done by equating the probability densities

$$P(x)=p(x) \tag{12}$$

and the weight function of the integrals (7) and (8)

$$\pi^{-}[A^{-}(x)]=p(x)^{q}, \quad \pi^{+}[A^{+}(x)]=p(x)^{q}. \tag{13}$$

By replacing (10) and (11) in the above expressions we find

$$\frac{dw^{-}[A^{-}(x)]}{dx}=p(x)^{q}, \quad \frac{dw^{+}[A^{+}(x)]}{dx}=-p(x)^{q} \tag{14}$$

Therefore,

$$w^{-}[A^{-}(x)]=\int_{-\infty}^{x} p_{q}(x')^{q}\,dx', \tag{15}$$

$$w^{+}[A^{+}(x)]=\int_{x}^{\infty} p_{q}(x')^{q}\,dx'. \tag{16}$$

Cumulative PT framework leaves room to w^{-} being other than w^{+}. Since in our case they are the same except by an additive constant and by the reversed signal, they result in the same decision weight. We will, therefore, focus on $w(A)\equiv w^{-}(A^{-})$.

The left side of each expression (15) and (16) is a function of A, while the right side is a functional of the $p_q(x)$. Both sides can be made consistent by defining a family of functions $w_q[A(x)]$. Each member of this family applies only to the Tsallis distribution of index q. Although there is not an explicit expression of w_q as a function of A, the value of w_q corresponding the value of A can be found for any given x, through equations (9) and

[2] The Cumulative PT was proposed originally in a discrete framework.

(15). We used that correspondence to plot figure 1, which shows some of these functions. In particular, the PT requirement that extreme events are overvalued is satisfied when $q \in [0,1]$. If $q \to 1$, then the expected utility theory is recovered.

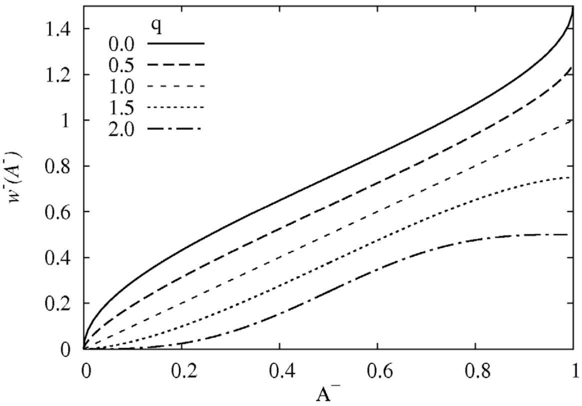

Figure 1. The capacity functions $w(A) \equiv w^-(A^-)$ for different values of q.

5 Conclusions

In this note, we have shown that the calculation of the expected values not from the probability but from a function of it is such unusual procedure that the correspondence between the PT and the Generalized Thermostatistics is unavoidable. Other peculiarity shared by both theories is the non normalization of the weighting function. However, distinctly to general PT, the functions $w_q[A(x)]$ derived from Tsallis Thermostatistics can only be used with the corresponding probability distribution, while the PT does not address such matters.

In particular, we have shown that the PT requirement that extreme events are overvalued is satisfied when $q \in [0,1]$. This is a very interesting result since it is the same range of validity of other behaviorial phenomenon found in Cajueiro (2005). In Cajueiro (2005), one of the authors has shown that the so-called intertemporal inconsistency may be represented for values of q belonging to the interval [0,1]. On the other hand, if $q \to 1$, then the expected utility theory is recovered and choices are consistent.

Acknowledgment

The authors are grateful to CNPQ for financial support and to an anonymous referee who made several useful recomendations.

References

ANTENEODO, C., TSALLIS, C. MARTINEZ, A. S. Risk aversion in economic transactions, *Europhysics Letters*, p. 635–641, 2002.

BLACK, F. SCHOLES, M. S. The pricing of options and corporate liabilities, *Journal of Political Economy*, p. 637–654, 1973.

BORGES, E. P. *Manifestações dinâmicas e termodinâmicas de sistemas não-extensivos*, Doutorado, Centro Brasileiro de Pesquisas Físicas, 2004.

BORLAND, L.. Option pricing formulas based on a non-gaussian stock price model, *Physical Review Letters*, 98701, 2002.

CAJUEIRO, D. O. A note on the relevance of the q-exponential function in the context of intertemporal choices, *Physica A*, p. 385–388, 2005.

KAHNEMAN, D.; TVERSKY, A. Prospect theory: An analysis of decisions under risk, *Econometrica*, p. 313–327, 1970.

MARTINEZ, A. S.; GONZÁLEZ, R. S.; TERÇARIOL, C. A. S. Continuous growth models in terms of generalized logarithm and exponential functions, *Physica A*, p. 5679–5687, 2008.

QUEIRÓS, S. M. D. On anomalous distributions in intra-day financial time series and non-extensive statistical mechanics, *Physica A*, p. 619–625, 2004.

TAKAHASHI, T. A comparison of intertemporal choices for oneself versus someone else based on tsallis' statistics, *Physica A*, p. 637–644, 2007.

TAKAHASHI, T. A probabilistic choice model based on tsallis' statistics, *Physica A*, p. 335–338, 2007.

TAKAHASHI, T. A comparison between tsallis's statistics-based and generalized quasi-hyperbolic discount models in humans, *Physica A*, p. 551–566, 2008.

TAKAHASHI, T. Tsallis' non-extensive free energy as a subjective value of an uncertain reward, *Physica A*, p. 715–719, 2009.

TAKAHASHI, T.; OONO, H.; RADFORD, M. H. B. Empirical estimation of consistency parameter in intertemporal choice based on tsallis' statistics, *Physica A*, p. 338–342. 2007.

TAKAHASHI, T.; OONO, H. RADFORD, M. H. B. Psychophysics of time perception and intertemporal choice models, *Physica A*, p. 2066–2074, 2008.

TSALLIS, C. Possible generalization of boltzmann-gibbs statistics, *Journal of Statistical Physics*, p. 479–487, 1988.

TSALLIS, C. *Introduction to nonextensive statistical mechanics – approaching a complex world*. Springer, 2009.

TSALLIS, C.; ANTENEODO, C.; BORLAND, L. et al. Nonextensive statistical mechanics and economics, *Physica A*, p. 89–100, 2003.

TVERSKY, A.; KAHNEMAN, D. Advances in prospect theory: cumulative representation of uncertainty. *Journal of Risk and Uncertainty*, p. 297–323, 1992.